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by

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IN
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Abstract

System reliability estimates are generally made using a model which assumes independence between components, and results are often claimed to be conservative. For a two component serial system bivariate distributions are developed for three cases: (1) bivariate exponential, (2) bivariate geometric, and (3) a composite exponential, geometric bivariate. These distributions are then utilized to investigate the reliability of a two component serial system when an estimate of the correlation coefficient is available. The estimate of the reliability thus obtained is then compared with the corresponding estimate obtained by use of the model which assumes that the system reliability is the product of the component reliabilities. The difference between these two estimates is tabulated for values of the correlation coefficient between -.25 and +.25, for each of the three bivariate distributions. Conditions under which the effect is maximum are explored and a method of approximating the reliability difference is suggested.

The writers wish to express their appreciation for the assistance and encouragement given them by Professor Walter Max Woods of the U. S. Naval Postgraduate School in this investigation.

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List of Tables and Illustrations

Table of Symbols and Abbreviations

Symbol	Definition
X	A random variable
Y	A random variable
T	A random variable denoting time
t_o	A particular time
k_o	A particular number of power turn-ons (PTO's)
$f_X(x;a)$	A probability function of the random variable X with parameter a (density function if X is continuous, mass function if X is discrete)
$F_X(x;a)$	A probability cumulative distribution function of the random variable X with parameter a
$P[\cdot]$	The probability of the event $[\cdot]$
$R(t)$	The reliability function evaluated at t
$R(t,k)$	The composite reliability function evaluated at time t and PTO's of k
$\hat{\rho}$	The estimated correlation coefficient
p	The probability of success on a Bernoulli trial
q	The probability of failure on a Bernoulli trial
v	The parameter of a class of derived bivariate distribution functions
$\chi^2_{N:\gamma}$	The probability $P[X \geq \gamma]$ where X has the chi-square distribution with N degrees of freedom
γ	A confidence coefficient, $0 \leq \gamma \leq 1$
ρ	The correlation coefficient between two variables
$\hat{a}$	An estimate of the parameter a

Abbreviation	
PTO	Power Turn-Ons (discrete random variable)
MLE	Maximum Likelihood Estimator
L.C.L.	Lower Confidence Limit
U.C.L.	Upper Confidence Limit
eq.	equation

Section 1

INTRODUCTION

It is known that various types of interdependence can exist between components of a general system. The resulting effects on the system reliability will be a function of the correlation between the components considered. A practice in reliability studies has been to assume that ignoring correlation effects would lead to a conservative estimate of the reliability. We propose here to study the validity of this assumption for various values of correlation and further to study quantitatively the resulting reliability estimates.

To narrow the scope of the problem, we have considered a serial system of two components and examined three distributions; (1) bivariate exponential; (2) bivariate geometric; and (3) a composite bivariate distribution where the marginals are exponential and geometric.

Each of these distributions is examined separately. A summary of their characteristics is given in Table 1.1. A comparison is made of the reliabilities defined by

$$R_1 (t) = P [X \geq t, Y \geq t] \equiv P [X \geq t] P [Y \geq t] \tag{1.1}$$

$$R_2 (t) = P [X \geq t, Y \geq t] \tag{1.2}$$

for positive and negative values of ρ.

The resulting estimates of reliability are analyzed and the effects of correlation on system reliability are evaluated. Exact confidence limits are derived where possible and approximations are considered otherwise.

Table 1.1

Probability Distributions

Exponential Family

Density : $f_T(t) = \begin{cases} \frac{1}{a} \exp(-t/a), & t \geq 0 \\ 0 & t < 0 \end{cases}$

Parameter : a $\qquad a > 0$

Mean : a

Variance : a^2

Geometric Family

Mass function: $p_K(k) = \begin{cases} p^k (1-p) & k = 1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases}$

Parameter : p $\qquad 0 \leq p \leq 1$

Mean : p/1-p

Variance : $p/(1-p)^2$

Bivariate Exponential

Density : $f_{X,Y}(x,y) = f_X(x)\ f_Y(y)\ [1 + v\ (2F_X(x)-1)\ (2F_Y(y)-1)]$

where

$$f_X(x) = \frac{1}{a} \exp(-x/a)$$

$$f_Y(y) = \frac{1}{b} \exp(-y/b)$$

$$F_X(x) = 1 - \exp(-x/a)$$

$$F_Y(y) = 1 - \exp(-y/b)$$

$x \geq 0$
$y \geq 0$
$-1 \leq v \leq 1$
$a \geq 0$
$b \geq 0$

Parameters : a, b, v

Section 2

GENERAL CONCEPTS OF RELIABILITY

2.1 Definition

Reliability, as the term is used in mathematical statistics, has exact meanings. It can be calculated, objectively evaluated, tested, and designed into equipment. A definition, as given by Lloyd and Lipow [1][1], is "the probability of a successful operation of the device in the manner and under the conditions of intended use". Mathematically then, reliability, at some point x_o, is the probability that a random variable, X, representing the operating life of some device, will equal or exceed the point x_o. It can be seen that if the probability distribution of the random variable is known, then the reliability at any point may be calculated.

A function representing the operating life may be either a continuous or discrete type of probability distribution. The interval of operation can be thought of as a time interval, in which case the probability distribution will be continuous, or as a number of operations (such as the turning on of a switch or relay) in which case the distribution is of the discrete type. Thus, for a continuous distribution the random variable T, to be considered, is the time to failure, while for a discrete distribution the random variable might be the number of power turn-ons (PTO's), K, before failure. Combinations of these are also possible. For example, the number of starts

1. Square brackets refer to correspondingly numbered references shown in Bibliography.

of a jet engine and the duration of operation may determine the life of the engine.

In the continuous case, the reliability, $R(t)$, of a component is the probability that the component will operate at least for some time, t. Thus, if $f_T(t)$ is the probability density function for T, then reliability or probability that the time to failure will exceed or equal t, is

$$R(t) = P\left[T \geq t\right] = \int_t^{\infty} f_T(t)dt \tag{2.1}$$

In the discrete case the life of the component may be measured by the number of PTO's prior to the first failure, where the random variable, say K, may take only discrete values 1, 2, 3, ... To determine the reliability at some fixed value k, where $R(k)$ = Probability [the number of PTO's prior to first failure will exceed k], we must determine the probability that K will take any value greater than k.

If the random variable K has a certain probability mass function $p(k)$ such that $0 \leq p(k) \leq 1$ and $\sum_{k=0}^{\infty} p(k) = 1$ where $p(k) = P\left[K = k\right]$, and the probability, p, that a certain single trial results in a success, is constant for any particular trial, ie, each PTO is a Bernoulli trial with fixed probability, p, of success, then the mass function is $p(k) = p^k(1 - p)$ (the probability of k successes and then a failure).

The reliability at some point k_o is then the probability that the random variable K will take any value greater than or equal to k_o.

Hence:

$$R(k_o) = \sum_{y=k_o}^{\infty} p^y q \qquad (2.2)$$

where $q = 1 - p$

Throughout this investigation we will assume that each trial does constitute a Bernoulli trial with fixed parameter p.

2.2 Component Interdependence

In determining the reliability of a system consisting of components, two basic configurations are of interest (1) a series system and (2) a parallel system. The series system consists of components put together in such a way that each component must operate for the system to operate, while in a parallel system the system will operate if any one component functions. This investigation will consider only a series system consisting of two components, with the probability distribution of each being known and the parameters of the distributions estimable for each.

If it is assumed that there is no component interaction in a series system, then the distributions are statistically independent and reliability of the system can be determined from the reliability of the components by multiplication of the component reliabilities. It should be noted, however, that the product of the component reliabilities may, under certain circumstances, indeed yield the system reliability even though strict independence does not hold. If there is interaction between components, then the product rule does not in

general hold. Rosenblatt [16] states that frequently the observation is made that "assessments of system reliability based on the assumption that component failures occur independently of one another are approximate and usually excessively conservative".

We will show that interaction between components can reduce as well as increase the reliability of a system. An example of interaction which reduces system reliability could be the case where two components, each of which produces heat when operated, causes a temperature environment which reduces the life of one or both components and hence of the system. Each component when operated separately, however, might produce less heat than is required to affect the life of that particular component. On the other hand in certain electronic devices a high temperature environment might be beneficial, in which case the interaction described would enhance the reliability of the system.

2.3 Statistical Estimation

In the estimation of reliability of a system from information available on the components, if the probability distribution family is known then the problem reduces to determining or estimating values of the parameters of the distribution. Two methods are available for doing this: (1) point estimates and (2) confidence interval estimates. A point estimate is the value of a statistic based on some experimental measurements. An example is the maximum likelihood estimate (M. L. E.), which has certain optimal qualities, see Mood [12].

A two-sided confidence interval estimate of a parameter is obtained by selecting two random values L and U such that, given a number $0 < \gamma < 1$, the statement that the random interval $[L, U]$ covers the parameter may be made with probability $1 - \gamma$. A one-sided confidence interval is obtained by selecting a single random value L where the corresponding confidence statement is, the interval $[L, \infty]$ covers the parameter with probability $1 - \gamma$.

The method by which experimental data is obtained is called the sampling plan. Several sampling plans are available for estimating parameters of the exponential distribution. Those considered in this investigation are:

Sampling Plan I: test N items of a given type until all fail and observe the N times to failure.

Sampling Plan II: test N items of a given type until r of them fail, where $r \leq N$ is fixed before starting the test. The observations are then the first r failure times.

Sampling Plan III: test n items of a given type to a preassigned time t_o. Let r denote the random number of failures in the specified time and observe the r failure times.

The only sampling plan which will be considered for the geometric case is as follows: observe the number r of failures for a pre-assigned number, N, of PTO's and any number of items of the given type. N is

fixed in advance of the test and r, the number of failures, is a random variable.

Association between two variables is estimated by several coefficients, the most notable of which are:

(i) Product moment correlation coefficient,

(ii) Spearman's rank correlation coefficient,

(iii) Kendall's τ correlation coefficient.

The performance of these correlation coefficients for general bivariate distributions are compared by Farlie [2].

2.4 Summary

In summary, our procedure for estimating reliability might be broken down into three steps: (1) to establish the type of statistical distribution which describes the failure phenomenon, here we assume this known, (2) to estimate the parameters which completely define the distribution, and (3) to utilize the knowledge of the distribution with the estimates of the parameters to estimate the reliability. In many present day applications a simplified model, the independent serial system model, in which the system reliability is calculated as the product of the reliability of the components, is used even though the actual reliability may be quite different and not always greater. It takes but little reflection to realize that in many applications an underestimate even by a small percentage might have great consequence on the cost of a large development program. It therefore seems very appropriate to attempt at least a start at a procedure by which the reliability can be estimated taking account of any component interaction.

Section 3

BIVARIATE EXPONENTIAL

3.1 Derivation

In this section we shall be concerned with developing a mathematical structure to represent the time to failure distribution of a two component system where each is exponentially distributed. Using this structure we shall determine the reliability of the system with known correlation and compare this with the product rule system reliability. We shall consider the marginal density functions to be of the form

$$f_T(t) = \frac{1}{a} \exp(-t/a), \quad t \geq 0. \tag{3.1}$$

The corresponding cumulative distrubution functions are of the form

$$F_T(t) = \int_0^t f_T(t)\, dt = 1 - \exp(-t/a). \tag{3.2}$$

Gumbel [3] has demonstrated a general method for deriving a bivariate distribution function and the corresponding density function from two known distribution functions by applying the formulas

$$F_{XY}(x,y) = F_X(x)\, F_Y(y) \left[1 + v\,(1 - F_X(x))\,(1 - F_Y(y))\right] \tag{3.3}$$

where $-1 \leqq v \leqq 1$ and consequently,

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) \left[1 + v\,(2F_X(x) - 1)\,(2F_Y(y) - 1)\right] \tag{3.4}$$

He has further derived two specific bivariate exponential distribution functions each of which is restricted to ranges of correlation less

than unity. We have chosen the more symmetric of the two distributions and have used it exclusively. For this case, where $F_X(x)$ and $F_Y(y)$ are both exponential, the bivariate exponential distribution and density functions become

$$F_{XY}(x,y)=\left[1-\exp(-x/a)\right]\left[1-\exp(-y/b)\right]\left[1+v\exp(-x/a-y/b)\right], \qquad (3.5)$$
$$x\geq 0,\ y\geq 0$$

and

$$f_{XY}(x,y)= \frac{1}{ab}\exp(-x/a-y/b)\Big[1+v\left[2\exp(-x/a)-1\right]\left[2\exp(-y/b)-1\right]\Big]. \qquad (3.6)$$

These can be shown (see Appendix A.1) to possess all the required properties and in particular the correlation coefficient can be expressed as

$$\rho = v/4. \qquad -1 \leq v \leq 1 \qquad (3.7)$$

From this it is seen that the correlation of this particular distribution is restricted to the range $-.25 \leq \rho \leq .25$.

3.2 System Reliability

As previously asserted, reliability is a probabilistic statement about the operating life of a unit. If we define reliability in symbols as

$$R(t) = P\left[T \geq t\right] \qquad (3.8)$$

then for the exponential case we have

$$R(t) = \int_t^{\infty} f_T(t) \; dt = \exp(-t/a). \tag{3.9}$$

If we now consider the reliability of a system of two exponentially distributed serial components, we express the system reliability as

$$R(t) = P\,[X \geq t,\ Y \geq t] = \int_t^{\infty}\int_t^{\infty} f_{XY}\,(x,y)\; dx\; dy \tag{3.10}$$

We seek to investigate the consequences of the assumption that

$$R(t) = R_1(t) \equiv P\,[X \geq t] \;\; P\,[Y \geq t] \tag{3.11}$$

when something is known of the correlation, ρ.

Using the product rule, eq. (3.11) is indeed true. However, for the general case the system reliability is found (see Appendix A.1) to be

$$R(t)=\exp(-t/a-t/b)\Big[1+v\big[1-\exp(-t/a)-\exp(-t/b)+\exp(-t/a-t/b)\big]\Big]\ . \tag{3.12}$$

This is monotone increasing in ρ and hence is restricted to the values of ρ between $-.25$ and $.25$. To obtain a more complete analysis the system reliability was derived for $\rho = 1$ by assuming that one variable was a linear function of the other, say $Y = cX$ where $c > 0$. In this special case the system reliability is given as

$$R(t) = P\Big[X \geq t,\ Y \geq t\Big] = \mathrm{Max}\Big[P\,\big[X \geq t\big],\ P\,\big[X \geq t/c\big]\Big] \tag{3.13}$$

since for $c \leq 1$, $R(t) = \exp(-t/a)$ and for $c > 1$, $R(t) = \exp(-t/ca)$.

To obtain a quantitative expression for the effect of correlation on the system reliability, a reliability difference function was constructed. To eliminate the need for considering both component parameters in this function we denote their ratio as $s = b/a$ and use this as a single parameter. Also, to avoid considering specific values of the operating time t, we shall use the ratio of the operating time to the mean time t/a as a parameter. With this notation we shall define the reliability difference function, denoted as $\Delta R(t)$, as the difference between the system reliability when there is correlation, $R_2(t)$, and the system reliability when there is no correlation, $R_1(t)$. Hence

$$\Delta R(t)=R_2(t)-R_1(t)=4\rho \exp\left[-t(s+1)/sa\right]\left[1-\exp(-t/a)-\exp(-t/sa)+\exp\left[-t(s+1)/sa\right]\right] \tag{3.14}$$

Equation (3.14) is valid for $-.25 \leq \rho \leq .25$, but for $\rho = 1$ it is

$$\Delta R(t) = \exp(-t/sa)\,[1 - \exp(-t/sa)]. \tag{3.15}$$

Values of the reliability difference functions defined in equations (3.14) and (3.15) are given in Tables 3.01 to 3.10 for values of the t/a = .05(.05)2.5 and s = .5(.5)5. Further, they are plotted in Figs. 3.1 and 3.2 with respect to t/a. The curves vary linearly with ρ up to .25 so only the curve for $\rho = .25$ is shown along with that for $\rho = 1$. To obtain values of $\Delta R(t)$ for ρ other than .25 the tables carry these for $\rho = .05$ and $\rho = .15$. At other values of ρ a linear interpolation will provide the answer.

For $s = 1$, when the component parameters are equal, the curve attains a maximum at $t/a = .69315$ and it is here that the effect of correlation on the system reliability is a maximum. When the parameters a and b are unequal, the curve peaks at different values of t/a depending on the value of s. For ρ less than .25, the maximum value of $\Delta R(t)$ is appreciably less than one-fourth that for $\rho = 1$ at any s other than one. The maximum value of $\Delta R(t)$ is .2500 for $\rho = 1$ and .0625 for $\rho = .25$ which is linear and consistent with previous knowledge.

Example 3.1

As an example let us compute the reliabilities and correlation effects for a system at time $t = 500$ hours where $a = b = 1000$ hours. Here $t/a = .5$ and the component reliabilities are $R(t) = \exp(-.5) = .60653$. The system reliability for the independent case is $R(t) = .36788$. From Table 3.02, $\Delta R(t)$ for $t/a = .5$ and $s = 1$ is .05695 for $\rho = .25$ and $\Delta R(t) = .23865$ for $\rho = 1.0$. For $\rho = +1.0$ the product rule underestimates the actual system reliability by 65%. Even for $\rho = +.25$ the underestimate is 15.5%.

3.3 Confidence Interval

A comprehensive discussion of the concepts and procedures in deriving estimates of the parameters and confidence intervals for the resulting reliability estimate is given in Chapters 7,8, and 10 of Lloyd and Lipow [1]. We shall consider three sampling plans which are most likely to be utilized in obtaining the failure data: (1) testing N items until all fail, (2) testing N items until $r < N$

fail, and (3) testing N items for time t_o and observing r, the number of failures. In each case the underlying distribution is exponential.

3.3.1 Testing N items independently until all fail

In this case, a sample of N items are put on test and the test is concluded when all have failed. The times to failure $t_1, \ldots, t_n$, each measured from the time the item was "turned on", are recorded. The intended life of the item is t, and it is required to demonstrate the reliability with confidence level $(1 - \gamma)$.

A maximum likelihood estimate for the parameter of the exponential distribution is

$$\hat{a} = \frac{1}{N} \sum_{i=1}^{N} t_i \quad (3.16)$$

It can be shown [1] that the function $C_a = 2N\hat{a}/a$ has the chi-square distribution with 2N degrees of freedom. Hence a lower confidence limit on R may be derived from the expression

$$P\left[2N\hat{a}/a > \chi^2_{2N:1-\gamma}\right] = 1-\gamma \quad (3.17)$$

This lower confidence limit, denoted by L , is

$$L = \exp\left(-t\left[\chi^2_{2N:1-\gamma}\right]/2N\hat{a}\right) \quad (3.18)$$

For the case of two independent exponential distributions $f_X(x)$ and $f_Y(y)$, if we define the ratio of the parameters as $s = b/a$ then, from (3.12), we can denote the reliability as

$$R(t) = \exp\left(-t(s+1)/sa\right). \quad (3.19)$$

Since $C_a = 2\,n_x\,\hat{a}/a$ has the chi-square distribution with $2\,n_x$ degrees of freedom and $C_b = 2\,n_y\,\hat{b}/b$ has the chi-square distribution with $2\,n_y$ degrees of freedom, let us define $C_{a,b} = 2\,n_x\,\hat{a}/a + 2\,n_y\,\hat{b}/b$. Then, by the reproductive property, $C_{a,b}$ has the chi-square distribution with $2\,n_x + 2\,n_y$ degrees of freedom. If we substitute $b = s\,a$, we get

$$C_{a,b} = 2\,(n_x\,s\,\hat{a} + n_y\,\hat{b})/s\,a. \tag{3.20}$$

Using the same analysis as for the univariate case, we see that

$$P\left[2(n_x s\hat{a} + n_y\,\hat{b})/sa > \chi^2_{2(n_x+n_y);\,1-\gamma}\right] = 1 - \gamma. \tag{3.21}$$

Hence a lower confidence limit for R is

$$L = \exp\left[-t(x+1)\,\chi^2_{2(n_x+n_y)\,;\,1-\gamma}\Big/2(n_x\,s\hat{a} + n_y\,\hat{b})\right] \tag{3.22}$$

with confidence level $1 - \gamma$.

3.3.2 Testing N items until $r < N$ fail

In this case a sample of N items are put on test and the test is concluded when some predetermined number $r \leq N$ have failed. The times to failure $t_1, \ldots, t_r$ are recorded where each t_i is measured from the time the item was "turned on". There arises the consideration of the immediate replacement of failed items. If replacement is used, the number on test is always N, whereas in the nonreplacement case, the number of items on test eventually drops to $N - r + 1$.

It can be shown [1] that in either the replacement or non-replacement case the quantity $(2\, r\, \hat{a}_{r,N})/a$ has the chi-square distribution with 2r degrees of freedom, where $\hat{a}_{r,N} = \sum_{i=1}^{r} t_i + (N-r)\, t_r$ Hence the procedures and results derived in section 3.3.1 can be directly applied, and we observe that

$$L = \exp\left[-t\chi^2_{2r\,;\,1-\gamma}/2r\, \hat{a}_{r,N}\right] \tag{3.23}$$

is the lower confidence limit on R(t) in the univariate case at confidence level $1 - \gamma$. Similarly, for the case of two independent exponential distributions a lower confidence limit on R(t) is

$$L = \exp\left[-t(s+1)\left(\chi^2_{2(r_x+r_y)\,;\,1-\gamma}\right)/2(sr_x\, \hat{a}_{r,N}+r_y\hat{b}_{r,N})\right] \tag{3.24}$$

at confidence level $1 - \gamma$.

3.3.3 Testing N items until time t_o

In this case we test N items for a predetermined time t_o. We note the number of failures, r, in this time and record the times to failure $t_1, \ldots, t_r$. These t_i are random variables with density function

$$f_T(t) = \frac{1}{A}\,\frac{1}{a}\,\exp(-t/a) \qquad 0 \le t \le t_o \tag{3.25}$$

where $A = 1 - \exp(-t_o/a)$. This is a density function since when integrated over the range of t the result is unity and $f_T(t)$ is non-negative. The MLE of a is then given by

$$\hat{a} = \frac{1}{r}\left[\sum_{i=1}^{r} t_i + (N-r)\, t_o\right] \tag{3.26}$$

hence an estimate of the reliability is

$$\hat{R}(t) = \exp(-t/\hat{a}). \tag{3.27}$$

In order to obtain a confidence interval, we note that the random variable r has a binomial distribution with parameters N and $p = 1 - \exp(-t_o/a) = A$. Hence we see that

$$L = (1 - \alpha(r))^{t/t_o} \tag{3.28}$$

where $\alpha(r)$ is the solution of $I_{1-\alpha(r)}[n-r, r+1] = \beta$ and $I_x(a,b)$ is the incomplete beta function tabulated by Karl Pearson.

For the bivariate case the procedures are extended in the previous manner. If we denote the estimators as $\hat{a}_x$ and $\hat{a}_y$ where

$$r_x\hat{a}_x = \sum_{i=1}^{r_x} t_{x_i} + (N-r_x)\, t_{x_o}$$

and

$$r_y\hat{a}_y = \sum_{i=1}^{r_y} t_{y_i} + (N-r_y)\, t_{y_o}$$

then we have

$$\hat{R}(t) = \exp[-t\,(1/\hat{a}_x + 1/\hat{a}_y)]. \tag{3.29}$$

Returning to our previous example, if we were to test 5 items of each component and we estimated the parameters as $\hat{a} = 975$ and $\hat{b} = 1050$ then a 95% lower confidence limit on R(t) would be

$$L = \exp(-500\,(\chi^2\ 20;.95)/10125) = .53585 \tag{3.30}$$

for the independent case. As a possible approach to a solution to

the dependent case we might try a procedure such as this: (i) obtain the independent confidence limit as above, (ii) by using one of the standard procedures derive an estimate for ρ, (iii) find $\Delta R(t)$ at t/a using the value of R and (iv) add algebraically this $\Delta R(t)$ to $L_{a,b}$ to obtain the final estimator. No claims are made as to the accuracy of this method, it is merely suggested as a possible means of obtaining a bound on reliability in the face of known correlation.

3.4 Approximating the Effect

If we examine the reliability difference function defined in section 3.2 and use the notation of eq. (3.12), we may rewrite eq. (3.14) as

$$\Delta R(t)=4\rho\exp(-t/a-t/b)\left[1-\exp(-t/a)-\exp(-t/b)+\exp(-t/a-t/b)\right]. \qquad (3.31)$$

By regrouping the terms slightly, we can put this into the form

$$\Delta R(t)=4\rho\exp(-t/a)\left(1-\exp(-t/a)\right)\exp(-t/b)\left(1-\exp(-t/b)\right) \qquad (3.32)$$

and we see immediately that this is merely 4ρ times the product of the component reliabilities and their unreliabilities for $-.25 \leq \rho \leq .25$.

As a check if we let $t/a = .5$ and $t/b = .25$ and $\rho = .25$ then $\Delta R(t) = .04111$ which checks with the result in Table 3.04 for $t/a = .5$ and $s = 2$. This was to be expected since in this particular case the approximation is identical to the exact method.

3.5 Summary

For a two component serial system the effect of correlation on system reliability is greatest at $t/a = .69315$ when the parameters are equal and the maximum $\Delta R(t)$ is .0625 for $\rho = .25$ and .2500 for $\rho = 1.0$. When the parameters are unequal, the point of maximum effect due to correlation varies directly as the ratio of the parameters. For the restricted bivariate distribution the maximum $\Delta R(t)$ is less than one fourth of that for the case $\rho = 1$ for values of s other than one, and this is likely due to the bivariate distribution used.

Lower confidence limits have been defined for the independent case under three sampling plans. For the dependent case an "ad hoc" procedure is suggested.

3.6 Description of Graph Format

At the end of this section and the succeeding sections are located the figures referred to in the text. These figures are presented in a standard format except for the axis scaling. This is indicated by the legend directly under the figure title. The following example illustrates the notation used.

<u>Example 3.2</u>

An axis scaling legend of

X AXIS SCALE = 2.00 E + 02
Y AXIS SCALE = 1.00 E - 02

is to be read as "the X AXIS is marked off in units of 2.00 X 10^{2} and the Y AXIS is marked off in units of 1.00 X 10^{-2}". A general legend would read "the X AXIS is marked off in units of A X 10^{B}" and be denoted as X AXIS SCALE = AE + B.

TABLE 3.01

TABLE OF EXPONENTIAL RELIABILITY DIFFERENCES

S= .500 S IS RATIO OF PARAMETERS B/A

T/A	RHO=.05	.15	.25	1.0
.05	.00080	.00240	.00399	.08611
.10	.00256	.00767	.01278	.14841
.15	.00460	.01381	.02302	.19201
.20	.00656	.01968	.03280	.22099
.25	.00822	.02467	.04111	.23865
.30	.00951	.02853	.04754	.24762
.35	.01040	.03121	.05202	.24999
.40	.01094	.03281	.05468	.24743
.45	.01115	.03345	.05575	.24127
.50	.01110	.03330	.05550	.23254
.55	.01084	.03252	.05420	.22207
.60	.01042	.03127	.05212	.21048
.65	.00989	.02968	.04947	.19826
.70	.00929	.02787	.04644	.18579
.75	.00864	.02592	.04320	.17334
.80	.00797	.02392	.03987	.16113
.85	.00731	.02192	.03654	.14931
.90	.00666	.01997	.03329	.13798
.95	.00603	.01810	.03017	.12720
1.00	.00544	.01633	.02721	.11702
1.05	.00489	.01467	.02445	.10746
1.10	.00438	.01313	.02188	.09853
1.15	.00390	.01171	.01952	.09021
1.20	.00347	.01042	.01736	.08249
1.25	.00308	.00924	.01540	.07535
1.30	.00273	.00818	.01363	.06876
1.35	.00241	.00722	.01204	.06269
1.40	.00212	.00637	.01061	.05711
1.45	.00187	.00560	.00934	.05200
1.50	.00164	.00492	.00820	.04731
1.55	.00144	.00432	.00719	.04302
1.60	.00126	.00378	.00630	.03910
1.65	.00110	.00331	.00551	.03552
1.70	.00096	.00289	.00482	.03226
1.75	.00084	.00252	.00420	.02929
1.80	.00073	.00220	.00367	.02658
1.85	.00064	.00192	.00320	.02411
1.90	.00056	.00167	.00278	.02187
1.95	.00048	.00145	.00242	.01983
2.00	.00042	.00126	.00210	.01798
2.05	.00037	.00110	.00183	.01630
2.10	.00032	.00095	.00159	.01477
2.15	.00028	.00083	.00138	.01338
2.20	.00024	.00072	.00119	.01213
2.25	.00021	.00062	.00104	.01099
2.30	.00018	.00054	.00090	.00995
2.35	.00016	.00047	.00078	.00901
2.40	.00013	.00040	.00067	.00816
2.45	.00012	.00035	.00058	.00739
2.50	.00010	.00030	.00050	.00669

TABLE 3.02

TABLE OF EXPONENTIAL RELIABILITY DIFFERENCES

S=1.000 S IS RATIO OF PARAMETERS B/A

T/A	RHC=.05	.15	.25	1.0
.05	.00043	.00129	.00215	.04639
.10	.00148	.00445	.00741	.08611
.15	.00287	.00862	.01437	.11989
.20	.00441	.01322	.02203	.14841
.25	.00594	.01781	.02968	.17227
.30	.00737	.02212	.03687	.19201
.35	.00866	.02598	.04331	.20810
.40	.00977	.02930	.04884	.22099
.45	.01068	.03203	.05339	.23106
.50	.01139	.03417	.05695	.23865
.55	.01191	.03574	.05957	.24408
.60	.01226	.03679	.06131	.24762
.65	.01245	.03735	.06226	.24951
.70	.01250	.03750	.06249	.24999
.75	.01242	.03727	.06212	.24924
.80	.01224	.03673	.06122	.24743
.85	.01198	.03594	.05989	.24473
.90	.01164	.03493	.05821	.24127
.95	.01125	.03375	.05625	.23717
1.00	.01082	.03245	.05408	.23254
1.05	.01035	.03105	.05175	.22748
1.10	.00986	.02959	.04931	.22207
1.15	.00936	.02809	.04682	.21638
1.20	.00886	.02658	.04430	.21048
1.25	.00836	.02507	.04179	.20442
1.30	.00786	.02358	.03931	.19826
1.35	.00738	.02213	.03688	.19203
1.40	.00690	.02071	.03452	.18579
1.45	.00645	.01934	.03224	.17955
1.50	.00601	.01803	.03005	.17334
1.55	.00559	.01677	.02796	.16720
1.60	.00519	.01558	.02596	.16113
1.65	.00482	.01445	.02408	.15517
1.70	.00446	.01338	.02229	.14931
1.75	.00412	.01237	.02061	.14358
1.80	.00381	.01142	.01904	.13798
1.85	.00351	.01054	.01756	.13251
1.90	.00324	.00971	.01618	.12720
1.95	.00298	.00894	.01489	.12203
2.00	.00274	.00822	.01369	.11702
2.05	.00252	.00755	.01258	.11216
2.10	.00231	.00693	.01155	.10746
2.15	.00212	.00635	.01059	.10292
2.20	.00194	.00582	.00971	.09853
2.25	.00178	.00533	.00889	.09429
2.30	.00163	.00488	.00814	.09021
2.35	.00149	.00447	.00744	.08627
2.40	.00136	.00408	.00680	.08249
2.45	.00124	.00373	.00622	.07885
2.50	.00114	.00341	.00568	.07535

TABLE 3.03

TABLE OF EXPONENTIAL RELIABILITY DIFFERENCES

S=1.500 S IS RATIO OF PARAMETERS B/A

T/A	RHC=.05	.15	.25	1.0
.05	.00029	.00088	.00147	.03171
.10	.00104	.00312	.00520	.06033
.15	.00206	.00619	.01032	.08611
.20	.00324	.00973	.01621	.10924
.25	.00448	.01343	.02239	.12995
.30	.00570	.01710	.02850	.14841
.35	.00686	.02058	.03430	.16480
.40	.00792	.02377	.03962	.17928
.45	.00887	.02662	.04436	.19201
.50	.00969	.02908	.04847	.20311
.55	.01038	.03115	.05192	.21274
.60	.01094	.03283	.05472	.22099
.65	.01138	.03413	.05689	.22799
.70	.01169	.03508	.05846	.23385
.75	.01190	.03569	.05948	.23865
.80	.01200	.03600	.06000	.24249
.85	.01201	.03604	.06007	.24546
.90	.01195	.03585	.05974	.24762
.95	.01181	.03544	.05907	.24905
1.00	.01162	.03486	.05809	.24982
1.05	.01137	.03412	.05687	.24999
1.10	.01109	.03326	.05543	.24961
1.15	.01076	.03229	.05382	.24874
1.20	.01042	.03125	.05208	.24743
1.25	.01005	.03014	.05023	.24572
1.30	.00966	.02898	.04831	.24366
1.35	.00927	.02780	.04633	.24127
1.40	.00887	.02660	.04433	.23860
1.45	.00846	.02539	.04232	.23568
1.50	.00806	.02419	.04031	.23254
1.55	.00766	.02299	.03832	.22921
1.60	.00727	.02182	.03637	.22571
1.65	.00689	.02067	.03446	.22207
1.70	.00652	.01956	.03259	.21830
1.75	.00616	.01847	.03079	.21443
1.80	.00581	.01742	.02904	.21048
1.85	.00547	.01641	.02736	.20645
1.90	.00515	.01545	.02574	.20238
1.95	.00484	.01452	.02419	.19826
2.00	.00454	.01363	.02272	.19411
2.05	.00426	.01278	.02131	.18995
2.10	.00399	.01198	.01996	.18579
2.15	.00374	.01122	.01869	.18162
2.20	.00350	.01049	.01749	.17747
2.25	.00327	.00981	.01634	.17334
2.30	.00305	.00916	.01527	.16924
2.35	.00285	.00855	.01425	.16517
2.40	.00266	.00798	.01329	.16113
2.45	.00248	.00743	.01239	.15714
2.50	.00231	.00693	.01154	.15320

TABLE 3.04

TABLE OF EXPONENTIAL RELIABILITY CIFFERENCES

S=2.CCO   S IS RATIC OF PARAMETERS B/A

T/A	RHC=.05	.15	.25	1.0
.05	.C0022	.00C67	.00112	.02408
.10	.C0080	.00240	.C0399	.04639
.15	.C0161	.00482	.C0804	.067C4
.20	.C0256	.00767	.01278	.C8611
.25	.C0357	.01C72	.01786	.10370
.30	.C0460	.01381	.02302	.11989
.35	.C0561	.01683	.02805	.13477
.40	.C0656	.01968	.03280	.14841
.45	.C0743	.02230	.03717	.16089
.50	.C0822	.02467	.04111	.17227
.55	.C0891	.02674	.04457	.18262
.60	.C0951	.02853	.04754	.19201
.65	.01000	.03001	.050C2	.20048
.70	.C1040	.03121	.05202	.20810
.75	.C1071	.03214	.05357	.21492
.80	.C1094	.03281	.C5468	.22C99
.85	.C11C8	.03324	.05540	.22635
.90	.C1115	.03345	.C5575	.23106
.95	.C1115	.03346	.C5577	.23514
1.00	.C1110	.03330	.05550	.23865
1.05	.C1099	.03298	.05496	.24162
1.10	.C1084	.03252	.C5420	.244C8
1.15	.C1065	.03195	.C5324	.24607
1.20	.C1042	.03127	.05212	.24762
1.25	.C1017	.03C51	.05085	.24876
1.30	.C0989	.02968	.04947	.24951
1.35	.C0960	.0288C	.04799	.24992
1.40	.C0929	.02787	.04644	.24999
1.45	.C0897	.02691	.04484	.24975
1.50	.C0864	.02592	.04320	.24924
1.55	.C0831	.02492	.04154	.24846
1.60	.C0797	.02392	.03987	.24743
1.65	.C0764	.02292	.03820	.24619
1.70	.C0731	.02192	.03654	.24473
1.75	.C0698	.02C94	.03490	.24309
1.80	.C0666	.01997	.03329	.24127
1.85	.C0634	.01903	.03171	.23929
1.90	.C06C3	.0181C	.03017	.23717
1.95	.C0573	.01720	.C2867	.23492
2.00	.C0544	.01633	.C2721	.23254
2.05	.CC516	.01548	.02580	.23006
2.10	.C0489	.01467	.02445	.22748
2.15	.C0463	.01388	.02314	.22481
2.20	.C0438	.01313	.02188	.22207
2.25	.C0413	.0124C	.02067	.21925
2.30	.C0390	.01171	.01952	.21638
2.35	.C0368	.01105	.01842	.21345
2.40	.C0347	.01C42	.C1736	.21048
2.45	.C0327	.00981	.C1636	.20746
2.50	.CC3C8	.00924	.C1540	.20442

TABLE 3.05

TABLE OF EXPONENTIAL RELIABILITY DIFFERENCES

S=2.500 S IS RATIO OF PARAMETERS B/A

T/A	RHC=.05	.15	.25	1.0
.05	.00018	.00054	.00090	.01941
.10	.00065	.00195	.00324	.03767
.15	.00132	.00395	.00658	.05484
.20	.00211	.00632	.01053	.07097
.25	.00297	.00890	.01483	.08611
.30	.00385	.01155	.01926	.10029
.35	.00473	.01418	.02364	.11357
.40	.00557	.01671	.02784	.12599
.45	.00636	.01908	.03179	.13759
.50	.00708	.02125	.03542	.14841
.55	.00774	.02321	.03868	.15848
.60	.00831	.02494	.04156	.16784
.65	.00881	.02643	.04405	.17653
.70	.00923	.02768	.04614	.18457
.75	.00957	.02871	.04786	.19201
.80	.00984	.02952	.04920	.19886
.85	.01004	.03012	.05021	.20515
.90	.01018	.03053	.05089	.21092
.95	.01026	.03077	.05128	.21619
1.00	.01028	.03083	.05139	.22099
1.05	.01025	.03076	.05126	.22534
1.10	.01018	.03055	.05091	.22925
1.15	.01007	.03022	.05037	.23276
1.20	.00993	.02979	.04965	.23589
1.25	.00976	.02927	.04879	.23865
1.30	.00956	.02868	.04779	.24107
1.35	.00934	.02802	.04669	.24315
1.40	.00910	.02730	.04550	.24493
1.45	.00885	.02655	.04424	.24641
1.50	.00858	.02575	.04292	.24762
1.55	.00831	.02494	.04156	.24856
1.60	.00803	.02410	.04016	.24926
1.65	.00775	.02325	.03875	.24972
1.70	.00746	.02239	.03732	.24996
1.75	.00718	.02154	.03589	.24999
1.80	.00689	.02068	.03447	.24982
1.85	.00661	.01984	.03306	.24948
1.90	.00633	.01900	.03167	.24895
1.95	.00606	.01818	.03030	.24827
2.00	.00579	.01737	.02895	.24743
2.05	.00553	.01659	.02764	.24645
2.10	.00527	.01582	.02636	.24534
2.15	.00502	.01507	.02512	.24410
2.20	.00478	.01435	.02392	.24274
2.25	.00455	.01365	.02275	.24127
2.30	.00432	.01297	.02162	.23970
2.35	.00411	.01232	.02054	.23804
2.40	.00390	.01169	.01949	.23629
2.45	.00370	.01109	.01849	.23445
2.50	.00350	.01051	.01752	.23254

TABLE 3.06

TABLE OF EXPONENTIAL RELIABILITY DIFFERENCES

S=3.000 S IS RATIO OF PARAMETERS B/A

T/A	RHO=.05	.15	.25	1.0
.05	.00015	.00045	.00075	.01626
.10	.00055	.00164	.00273	.03171
.15	.00111	.00334	.00556	.04639
.20	.00179	.00537	.00895	.06033
.25	.00253	.00760	.01267	.07356
.30	.00331	.00992	.01653	.08611
.35	.00408	.01224	.02039	.09799
.40	.00483	.01449	.02414	.10924
.45	.00554	.01662	.02770	.11989
.50	.00620	.01861	.03101	.12995
.55	.00681	.02042	.03404	.13945
.60	.00735	.02205	.03675	.14841
.65	.00783	.02348	.03914	.15685
.70	.00824	.02472	.04120	.16480
.75	.00859	.02576	.04294	.17227
.80	.00887	.02662	.04436	.17928
.85	.00910	.02729	.04548	.18585
.90	.00927	.02780	.04633	.19201
.95	.00938	.02814	.04690	.19775
1.00	.00945	.02834	.04723	.20311
1.05	.00947	.02840	.04734	.20810
1.10	.00945	.02835	.04724	.21274
1.15	.00939	.02818	.04696	.21703
1.20	.00930	.02791	.04651	.22099
1.25	.00918	.02755	.04592	.22464
1.30	.00904	.02712	.04520	.22799
1.35	.00887	.02662	.04437	.23106
1.40	.00869	.02607	.04345	.23385
1.45	.00849	.02546	.04244	.23638
1.50	.00827	.02482	.04137	.23865
1.55	.00805	.02415	.04024	.24069
1.60	.00781	.02344	.03907	.24249
1.65	.00757	.02272	.03787	.24408
1.70	.00733	.02199	.03665	.24546
1.75	.00708	.02125	.03541	.24663
1.80	.00683	.02050	.03417	.24762
1.85	.00658	.01975	.03292	.24842
1.90	.00634	.01901	.03168	.24905
1.95	.00609	.01827	.03045	.24951
2.00	.00585	.01754	.02923	.24982
2.05	.00561	.01682	.02804	.24998
2.10	.00537	.01612	.02686	.24999
2.15	.00514	.01543	.02571	.24986
2.20	.00492	.01476	.02459	.24961
2.25	.00470	.01410	.02350	.24924
2.30	.00449	.01346	.02244	.24874
2.35	.00428	.01284	.02141	.24814
2.40	.00408	.01225	.02041	.24743
2.45	.00389	.01167	.01945	.24662
2.50	.00370	.01111	.01851	.24572

TABLE 3.07

TABLE OF EXPONENTIAL RELIABILITY DIFFERENCES

S=3.500 S IS RATIO OF PARAMETERS B/A

T/A	RHO=.05	.15	.25	1.0
.05	.00013	.00039	.00065	.01398
.10	.00047	.00141	.00236	.02737
.15	.00096	.00289	.00482	.04019
.20	.00156	.00467	.00779	.05246
.25	.00221	.00663	.01106	.06418
.30	.00290	.00869	.01448	.07540
.35	.00358	.01075	.01792	.08611
.40	.00426	.01277	.02129	.09633
.45	.00490	.01471	.02451	.10609
.50	.00551	.01652	.02754	.11540
.55	.00607	.01820	.03033	.12427
.60	.00657	.01972	.03286	.13272
.65	.00702	.02107	.03512	.14076
.70	.00742	.02226	.03710	.14841
.75	.00776	.02328	.03880	.15568
.80	.00805	.02414	.04023	.16258
.85	.00828	.02483	.04139	.16913
.90	.00846	.02538	.04230	.17533
.95	.00860	.02579	.04298	.18120
1.00	.00869	.02606	.04343	.18676
1.05	.00874	.02621	.04368	.19201
1.10	.00875	.02624	.04374	.19696
1.15	.00873	.02618	.04363	.20162
1.20	.00867	.02602	.04336	.20601
1.25	.00859	.02577	.04295	.21013
1.30	.00849	.02546	.04243	.21400
1.35	.00836	.02507	.04179	.21761
1.40	.00821	.02463	.04106	.22099
1.45	.00805	.02415	.04024	.22414
1.50	.00787	.02362	.03936	.22707
1.55	.00768	.02305	.03842	.22978
1.60	.00749	.02246	.03743	.23229
1.65	.00728	.02184	.03640	.23460
1.70	.00707	.02121	.03534	.23672
1.75	.00685	.02056	.03426	.23865
1.80	.00663	.01990	.03317	.24041
1.85	.00641	.01924	.03207	.24200
1.90	.00619	.01858	.03096	.24343
1.95	.00597	.01792	.02986	.24469
2.00	.00575	.01726	.02876	.24581
2.05	.00554	.01661	.02768	.24678
2.10	.00532	.01597	.02661	.24762
2.15	.00511	.01533	.02556	.24832
2.20	.00490	.01471	.02452	.24889
2.25	.00470	.01411	.02351	.24933
2.30	.00450	.01351	.02252	.24966
2.35	.00431	.01293	.02156	.24988
2.40	.00412	.01237	.02062	.24999
2.45	.00394	.01183	.01971	.24999
2.50	.00377	.01130	.01883	.24989

TABLE 3.08

TABLE OF EXPONENTIAL RELIABILITY DIFFERENCES

S=4.000 S IS RATIO OF PARAMETERS B/A

T/A	RHC=.05	.15	.25	1.0
.05	.00011	.00034	.00057	.01227
.10	.00041	.00124	.00207	.02408
.15	.00085	.00255	.00425	.03545
.20	.00138	.00413	.00689	.04639
.25	.00196	.00588	.00980	.05692
.30	.00257	.00772	.01287	.06704
.35	.00319	.00958	.01597	.07676
.40	.00381	.01142	.01903	.08611
.45	.00439	.01318	.02197	.09508
.50	.00495	.01485	.02475	.10370
.55	.00547	.01640	.02733	.11196
.60	.00594	.01781	.02969	.11989
.65	.00636	.01909	.03181	.12749
.70	.00674	.02021	.03369	.13477
.75	.00707	.02120	.03533	.14174
.80	.00734	.02203	.03672	.14841
.85	.00758	.02273	.03788	.15479
.90	.00776	.02329	.03882	.16089
.95	.00791	.02372	.03954	.16671
1.00	.00801	.02404	.04006	.17227
1.05	.00808	.02424	.04039	.17757
1.10	.00811	.02433	.04055	.18262
1.15	.00811	.02433	.04056	.18743
1.20	.00808	.02425	.04041	.19201
1.25	.00803	.02408	.04014	.19635
1.30	.00795	.02385	.03975	.20048
1.35	.00785	.02355	.03925	.20440
1.40	.00773	.02320	.03866	.20810
1.45	.00760	.02280	.03799	.21161
1.50	.00745	.02235	.03726	.21492
1.55	.00729	.02187	.03646	.21805
1.60	.00712	.02137	.03561	.22099
1.65	.00694	.02083	.03472	.22376
1.70	.00676	.02028	.03380	.22635
1.75	.00657	.01971	.03285	.22879
1.80	.00638	.01913	.03188	.23106
1.85	.00618	.01854	.03090	.23318
1.90	.00598	.01795	.02991	.23514
1.95	.00578	.01735	.02892	.23697
2.00	.00559	.01676	.02793	.23865
2.05	.00539	.01616	.02694	.24020
2.10	.00519	.01558	.02596	.24162
2.15	.00500	.01500	.02500	.24291
2.20	.00481	.01443	.02405	.24408
2.25	.00462	.01387	.02311	.24513
2.30	.00444	.01332	.02220	.24607
2.35	.00426	.01278	.02130	.24690
2.40	.00409	.01226	.02043	.24762
2.45	.00391	.01174	.01957	.24824
2.50	.00375	.01125	.01874	.24876

TABLE 3.09

TABLE OF EXPONENTIAL RELIABILITY DIFFERENCES

S=4.5CC S IS RATIO OF PARAMETERS B/A

T/A	RHC=.05	.15	.25	1.C
.05	.C0010	.00C3C	.00051	.01093
.10	.00037	.00111	.C0185	.02149
.15	.C0076	.00228	.00380	.03171
.20	.C0123	.00370	.00617	.04158
.25	.C0176	.00528	.00881	.05112
.30	.C0232	.00695	.01158	.06C33
.35	.C0288	.00864	.01441	.06923
.40	.C0344	.01C32	.01720	.07782
.45	.C0398	.01194	.01990	.08611
.50	.C0449	.01347	.02246	.09410
.55	.CC497	.01491	.02485	.10181
.60	.00541	.01623	.02705	.10924
.65	.C0581	.01743	.02905	.11641
.70	.C0617	.0185C	.03083	.12331
.75	.C0648	.01943	.03239	.12995
.80	.C0675	.02024	.03374	.13634
.85	.C0697	.02C92	.03487	.14250
.90	.C0716	.02148	.03581	.14841
.95	.C0731	.02193	.03655	.15410
1.00	.00742	.02226	.0371C	.15956
1.05	.C0750	.02249	.03749	.16480
1.10	.C0754	.02263	.03771	.16983
1.15	.C0756	.02268	.03779	.17466
1.20	.C0755	.02264	.03773	.17928
1.25	.C0751	.02253	.03755	.18371
1.30	.C0745	.02236	.03726	.18795
1.35	.C0737	.02212	.03687	.192C1
1.40	.C0728	.02184	.03639	.19588
1.45	.00717	.02150	.03583	.19958
1.50	.C0704	.02113	.C3521	.20311
1.55	.C0690	.02071	.03452	.20648
1.60	.C0676	.02C27	.03379	.20969
1.65	.C0660	.01981	.03301	.21274
1.70	.C0644	.01932	.03220	.21563
1.75	.C0627	.01881	.03135	.21838
1.80	.C0610	.01829	.03049	.22C99
1.85	.C0592	.01777	.02961	.22346
1.90	.C0574	.01723	.02872	.22579
1.95	.C0556	.01669	.02782	.22799
2.00	.C0538	.01615	.02692	.23CC7
2.05	.C0520	.01561	.02602	.23202
2.10	.C0503	.01508	.02513	.23385
2.15	.C0485	.01455	.C2424	.23556
2.20	.C0467	.01402	.02337	.23716
2.25	.C0450	.0135C	.0225C	.23865
2.30	.C0433	.01299	.02165	.24CC3
2.35	.C0416	.01249	.02082	.24131
2.40	.CC4C0	.0120C	.C2000	.24249
2.45	.C0384	.01152	.01921	.24357
2.50	.C0369	.01106	.01843	.24456

TABLE 3.10

TABLE OF EXPONENTIAL RELIABILITY DIFFERENCES

S=5.000 S IS RATIO OF PARAMETERS B/A

T/A	RHO=.05	.15	.25	1.0
.05	.00009	.00027	.00046	.00985
.10	.00033	.00100	.00167	.01941
.15	.00069	.00206	.00344	.02868
.20	.00112	.00335	.00559	.03767
.25	.00160	.00480	.00799	.04639
.30	.00211	.00632	.01053	.05484
.35	.00262	.00787	.01312	.06304
.40	.00314	.00941	.01568	.07097
.45	.00364	.01091	.01818	.07866
.50	.00411	.01233	.02055	.08611
.55	.00456	.01367	.02278	.09332
.60	.00497	.01490	.02483	.10029
.65	.00534	.01603	.02671	.10704
.70	.00568	.01704	.02839	.11357
.75	.00598	.01793	.02988	.11989
.80	.00624	.01871	.03118	.12599
.85	.00646	.01937	.03228	.13189
.90	.00664	.01992	.03320	.13759
.95	.00679	.02036	.03394	.14310
1.00	.00690	.02071	.03451	.14841
1.05	.00699	.02096	.03493	.15354
1.10	.00704	.02112	.03519	.15848
1.15	.00706	.02119	.03532	.16325
1.20	.00707	.02120	.03533	.16784
1.25	.00704	.02113	.03522	.17227
1.30	.00700	.02100	.03500	.17653
1.35	.00694	.02081	.03469	.18063
1.40	.00686	.02057	.03429	.18457
1.45	.00676	.02029	.03382	.18837
1.50	.00666	.01997	.03328	.19201
1.55	.00654	.01961	.03269	.19550
1.60	.00641	.01923	.03204	.19886
1.65	.00627	.01881	.03135	.20207
1.70	.00613	.01838	.03063	.20515
1.75	.00598	.01793	.02988	.20810
1.80	.00582	.01746	.02910	.21092
1.85	.00566	.01698	.02831	.21362
1.90	.00550	.01650	.02750	.21619
1.95	.00534	.01601	.02668	.21865
2.00	.00517	.01552	.02586	.22099
2.05	.00501	.01502	.02504	.22322
2.10	.00484	.01453	.02421	.22534
2.15	.00468	.01404	.02340	.22735
2.20	.00452	.01355	.02259	.22925
2.25	.00436	.01307	.02179	.23106
2.30	.00420	.01260	.02100	.23276
2.35	.00404	.01213	.02022	.23437
2.40	.00389	.01167	.01946	.23589
2.45	.00374	.01123	.01871	.23732
2.50	.00360	.01079	.01798	.23865

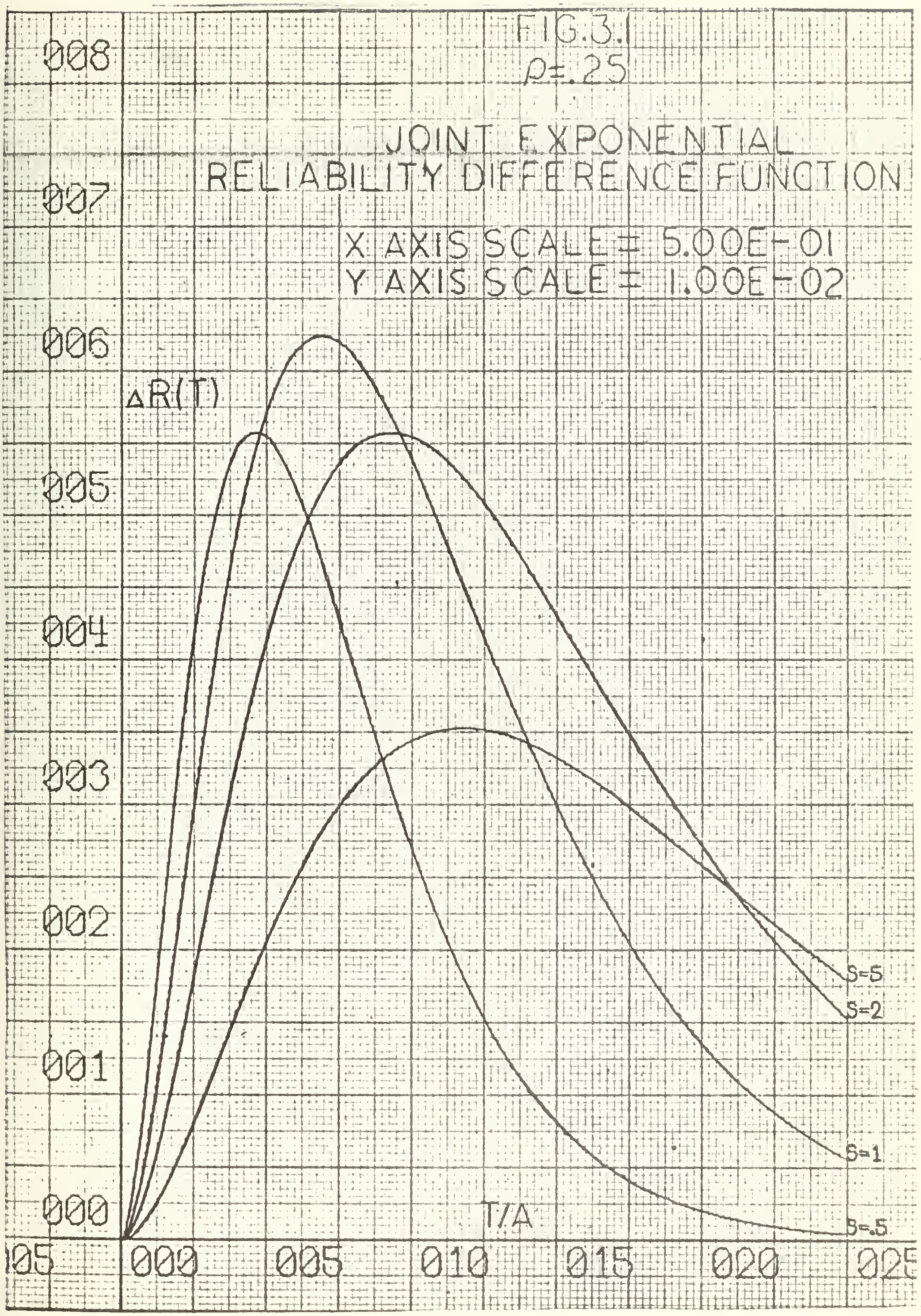

FIG.3.1
ρ=.25
JOINT EXPONENTIAL
RELIABILITY DIFFERENCE FUNCTION
X AXIS SCALE = 5.00E-01
Y AXIS SCALE = 1.00E-02
ΔR(T)
T/A
S=5
S=2
S=1
S=.5
008
007
006
005
004
003
002
001
000
005
000
005
010
015
020
025

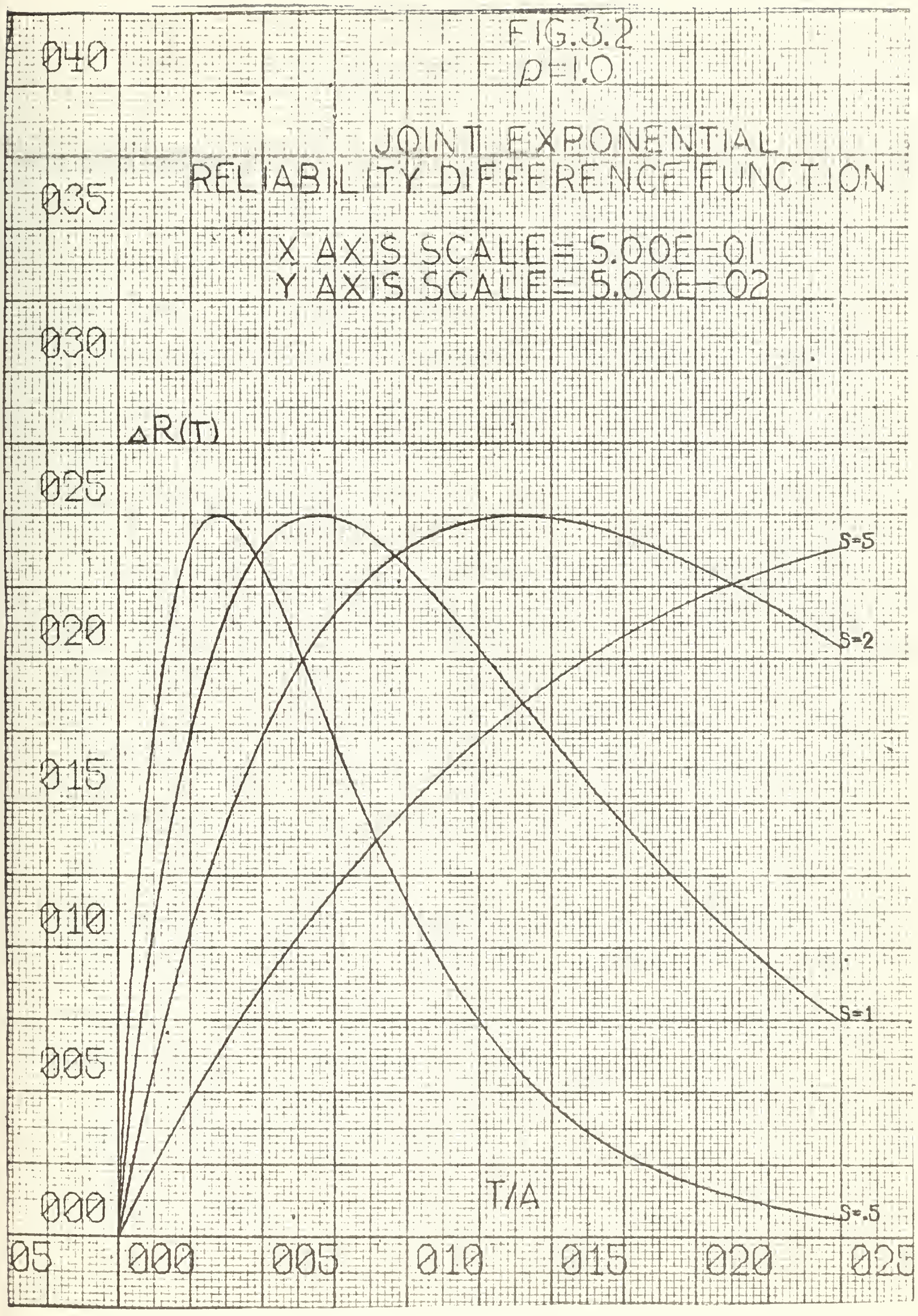
FIG.3.2
ρ=1.0
JOINT EXPONENTIAL
RELIABILITY DIFFERENCE FUNCTION
X AXIS SCALE=5.00E-01
Y AXIS SCALE=5.00E-02
ΔR(T)
T/A
S=5
S=2
S=1
S=.5
040
035
030
025
020
015
010
005
000
05
000
005
010
015
020
025

Section 4

BIVARIATE GEOMETRIC DISTRIBUTION

4.1 Derivation

In this section we shall develop a discrete bivariate distribution based on a general class of joint distributions. We shall take as marginals the geometric distribution and then evolve a relation from which the reliability of a two component system may be determined. Using this relation we shall then examine the effect on the reliability resulting from various values of correlation between the two components.

To investigate the effect of interdependence of two components when the probability of successful operation of a component follows a discrete distribution, a model applicable to "power turn-ons" was used. In this model the number of power-turn-ons prior to the first failure is considered a random variable, X. It is assumed that the probability of a success (ie, a switch functions properly, a relay opens, etc.) is a constant, p. The probability of a failure, q, is then (1-p). The reliability at k PTO's R(k), is then defined as the probability of at least k successes before the first failure. The random variable X then follows a geometric distribution: $p_X(x) = p^x q$ where the cumulative distribution is given by:

$$P_X(x) = P\left[X \geq x\right] = \sum_{i=0}^{x} p^i q = 1 - p^{x+1} \tag{4.1}$$

(see Appendix A.2)

The reliability of the component, ie, the probability that the random variable X will equal or exceed some constant, k_o, is given by

$$R(k_o) = P\left[X \geq k_o\right] = 1 - P\left[X \leq k_o - 1\right] = 1 - \left[1 - p^{k_o}\right] = p^{k_o} \tag{4.2}$$

In order to study two components, each having a probability distribution as defined above, a joint distribution, having identical but not necessarily independent, geometric distributions as marginals, was needed. Such a joint distribution is not unique but a class of such functions was determined by D. J. G. Farlie [2] in order to compare various correlation coefficients. (This investigation will consider only the product moment correlation coefficient, which is the more well known of the various types).

The joint mass function:

$$p_{XY}(x,y) = p_X(x)p_Y(y)\left[1+v\left[2P_X(x)-p_X(x)-1\right]\left[2P_Y(y)-p_Y(y)-1\right]\right] \tag{4.3}$$

$$x = 0, 1, 2, \ldots$$

$$y = 0, 1, 2, \ldots$$

was used. It can be shown (see Appendix A.2) that

(1) $p_{XY}(x,y) \geq 0$

(2) $\sum_{x=0}^{\infty} \sum_{y=0}^{\infty} p_{XY}(x,y) = 1$

(3) $\sum_{x=0}^{\infty} p_{XY}(x,y) = p_Y(y)$ and $\sum_{y=0}^{\infty} p_{XY}(x,y) = p_X(x)$

Also it can be seen that when $v = 0$ the joint distribution reduces to the product of the marginal distributions. Therefore, this $p_{XY}(x,y)$ does meet the conditions for a discrete bivariate distribution function.

To determine the functional relation between the constant v in $p_{XY}(x,y)$ and the product moment correlation coefficient, ρ, the defining equation for ρ was used.

$$\rho = \frac{E\left[X,Y\right] - E\left[X\right] E\left[Y\right]}{\sigma_X \ \sigma_Y} \tag{4.4}$$

Using p_1 for the parameter of the first component, ie, the component characterized by the random variable X, and p_2 for the parameter of the second component, the value of the correlation coefficient, ρ , was determined as:

$$\rho = \frac{v \sqrt{p_1 \ p_2}}{(1+p_1) \ (1+p_2)} \tag{4.5}$$

(calculations are shown in Appendix A.2)

From this relationship it can be seen that the values of ρ vary from 0 when p_1 or p_2 are zero to a maximum of $\frac{v}{4}$ when $p_1 = p_2 = 1$. Now since $-1 \leq v \leq 1$, which implies that $-\frac{1}{4} \leq \rho \leq \frac{1}{4}$, and hence this joint distribution function is satisfactory as a method by which to examine the effects on the reliability of the components having a correlation within this range.

With this joint mass function, an equation for the reliability $R(k_o) = P[X \geq k_o, Y \geq k_o]$, was developed (see Appendix A.2). Using the previously developed relation between ρ and v, the reliability as a function of the correlation is:

$$R_2(k_o) = p_1^{k_o} p_2^{k_o} + \frac{\rho(1+p_1)(1+p_2)}{\sqrt{p_1 p_2}} \left[\left[p_1^{k_o} - p_1^{2k_o}\right] \left[p_2^{k_o} - p_2^{2k_o}\right] \right] \qquad (4.6)$$

If it is assumed that the product rule holds, then the system reliability is the product of the component reliabilities.

$$R_1(k_o) = p_1^{k_o} \; p_2^{k_o} = (p_1 p_2)^{k_o} \qquad (4.7)$$

The difference between the two reliabilities, $\Delta R(k_o)$, is then the difference $R_2(k_o) - R_1(k_o)$.

4.2 System Reliability

To study the reliability difference the functional relation for $\Delta R(k_o)$ was determined.

$$\Delta R(k_o) = \frac{\rho(1+p_1)(1+p_2)}{\sqrt{p_1 p_2}} \left[\left[p_1^{k_o} - p_1^{2k_o}\right] \left[p_2^{k_o} - p_2^{2k_o}\right] \right] \qquad (4.8)$$

Using eq. 4.8 and assuming $p_1 = p_2$, the derivative with respect to k_o of $\Delta R(k_o)$ was set equal to zero in order to determine the value of k_o at which the function attained a relative maximum (development shown in Appendix A.2). The value of k_o at which this maximum occurs is

$$k_o = - \frac{\ln 2}{\ln p} \tag{4.9}$$

For $p = .999$ this gives a k_o for maximum difference of 693. Taking the mean life of a component as $\frac{p}{q}$ and taking the ratio of k_o to this mean life, gives $k_o/p/q = .693 \simeq .7$. Note that for the case of the bivariate exponential the corresponding point was $t/a = .69315$. That is, the ratio of t, the point at which the effect on reliability due to correlation is a maximum, to a, the mean life of a component, is constant and is approximately equal to .7.

This relation, (4.8), was programmed using FORTRAN language and the CDC 1604 computer, and is tabled in Tables 4.01 through 4.15.

It can be seen that $\Delta R(k_o)$ is monotone increasing in ρ, and hence the value of the reliability will be increased when ρ is positive and will be decreased when ρ is negative. All tables are computed for only positive values of ρ but because of the symmetry, the values are also good for negative ρ; that is

$[\Delta R(k_o, -\rho) = -\Delta R(k_o, \rho)]$.

There is a separate table for each combination of values of p_1 and p_2, for values .995(.001).999. Tables were made for values of k from 25(25)1000 and for ρ of .05(.05).25. The ratio of p_1 to p_2 is designated by s and is shown for each table. Only values of $p_1 \leq p_2$ are used since for any particular case the component having the smaller value of p may be designated 1.

Using the tabulated values, curves were plotted (utilizing the CDC 1604 computer) for $p_1 = .995$ and $p_2 = .995(.001).999$, each graph depicts the five values of the correlation coefficient, ρ. It can be seen that the magnitude of the effect increases as the values of the parameters approach a single value and reaches a maximum, when $p_1 = p_2$, of .062494 for $\rho = .25$. Notice that this maximum value is approximately .25 ρ, which is ρ times the maximum effect due to correlation. (Maximum effect given by Lloyd and Lipow [1]).

There is an interesting association between the value of k at which the maximum occurs and the "mean" of the system. If we define a system mean as:

$$m = \frac{\left(\frac{p_1 + p_2}{2}\right)}{1 - \left(\frac{p_1 + p_2}{2}\right)} \tag{4.10}$$

and if k^* is the point at which the effect is maximum, then the ratio k^*/m is constant and has the approximate value .7. For the case where $p_1 = p_2$ the ratio has already been shown to hold (note that for that case the mean just defined does in fact reduce to the mean of a component).

<u>Example 4.1</u>

As another example, if we take $p_1 = .997$, $p_2 = .999$, then $m = 499$. Now from Table 4.03 it can be seen that for all values of ρ the function is a maximum at $k = 350$, and therefore k/m is very close to .7.

Example 4.2

If we take the case where $p_1 = .995$, $p_2 = .998$, then $k = 284$ and from Table 4.06 $\Delta R(k)$ is maximum when $k = 200$, this gives $k/m = .704$.

This was done for all values of p_1 and p_2 which are tabled and the value of k/m was in each case $= .7 \pm .01$.

This relationship might be useful for design purposes in determining the component parameter values which would best utilize an enhancing interaction between two components, or conversely, specifying parameter values away from these if the interaction is degrading to the reliability.

4.3 Confidence Interval

To determine a confidence interval for the reliability, the independent model was used to first obtain a confidence interval when there is no interaction. The method used is that given in Lloyd and Lipow [1] page 226. Although other methods are available, see Buehler [11], Steck [13], and Madansky [14], there seems to be no generally accepted best method and therefore the procedure used was picked because of the ease with which it could be applied.

The procedure for a two component system is to compute

$$\hat{P} = \left(\frac{N_1 - f_1}{N_1}\right)\left(\frac{N_2 - f_2}{N_2}\right) \qquad (4.11)$$

and the quantity $N_m (1 - \hat{P}) = F$.

Where N_i is the number of trials of the i^{th} component and f_i is the number of failures of the same component, N_m is the minimum N_i. The number, F, is then considered to be the number of system failures in N_m trials of the system. With these as arguments the graphs given in [1] page 498 - 502 are utilized to obtain a lower confidence limit for any chosen confidence coefficient γ.

Example 4.3

If we take $N_1 = N_2 = 1000$, $\gamma = .95$, $f_1 = f_2 = 1$, $k = 50$, then $\hat{P} = (.999)\cdot(.999) = .998001$, $N_m = 1000$, $F = 2$. There results a 95% lower confidence limit on $\hat{P}$ of .994. Now since the reliability at $k = 50$ is given by

$$R(50) = (p_1 \; p_2)^{50} \tag{4.12}$$

and a lower confidence limit on $(p_1 \; p_2)$ is given by the lower confidence limit $\hat{P}$; then if $\hat{R}$ is a lower confidence limit on R, the 95% L.C.L. for this example is:

$$\hat{R}_o = (\hat{P})^{50} = (.994)^{50} \simeq .74 \tag{4.13}$$

Bounds on $\Delta R(k)$ for all values of ρ are $\pm$ .25 (For proof of this statement see [1] page 223) and the limit is attained only when $\rho = \pm 1$. Since ρ is, however, not known but must be estimated, there would also be a confidence interval associated with the estimate.

A lower confidence limit on R(k) for the dependent model could then be given by

$$\hat{R}(k) = \hat{R}_o - .25 \tag{4.14}$$

For Example 4.3 the 95% L.C.L. on R(k) would then be $\hat{R}(50) = .74-.25 = .49$ If, however, it were definitely known that ρ was positive, then the L.C.L., $\hat{R}(k)$ would be .74 since in that case any effect due to correlation would be enhancing.

It is recognized that such an L.C.L. on the reliability would not be "good", in the sense of shortest interval, and that perhaps a much better technique could be found.

4.4 Approximating the Effect

Looking at the relationship obtained for $\Delta R(k_o)$ given by equation (4.8) and examining it part by part, we see that:

$$\frac{\rho(1 + p_1)(1 + p_2)}{\sqrt{p_1 p_2}} \simeq 4\rho \tag{4.15}$$

for values of p_1 and p_2 close to 1.

Taking the next part of equation (4.8)

$$p^k - p^{2k} = p^k (1 - p^k) \tag{4.16}$$

It can be seen that this is the product of the reliability and the unreliability. Now since the third part of (4.8) is of the same form as the second part, an approximation for the difference function can be given by:

$$\Delta R(k_o) = 4 \rho \; p_1^{k_o} \; (1 - p_1^{k_o}) \; p_2^{k_o} \; (1 - p_2^{k_o}) \tag{4.17}$$

Notice that this is 4 ρ times the product of the reliability and the unreliability of each component.

Example 4.4

As an example of the use of this approximate method, let $p_1 = p_2 = .999$ and $\rho = .25$, $k_o = 100$. Then using the approximation equation (4.17), the reliability difference is : $\Delta R(k_o) = .007523$. Now using Table 4.05, the value given for the same conditions is .007421.

Example 4.5

As another example where this time $p_1 \neq p_2$, let $p_1 = .995$, $p_2 = .998$, $k_o = 200$ and $\rho = .10$. For this case the approximation yields $\Delta R(k_o) = .020568$. Using Table 4.06, a value of $\Delta R(k_o) =$ .020543 is obtained.

It can thus be seen that as a fast approximation the method suggested by equation (4.17) does yield good results.

4.5 Summary

In this section a jointly discrete distribution, a bivariate geometric, was developed from a large general class of distributions. It was shown that the particular distribution does in fact meet the requirements for a probability distribution function, however, no claim is made to its being unique. Using this distribution the effect of correlation was examined for a limited range of values of the product moment correlation coefficient. Specifically the difference between the reliabilities when using an estimated correlation

coefficient and when the independent model is assumed, was examined. Values of the reliability difference were computed and are tabulated. A graphical comparison is presented and indicates (1) that the maximum reliability difference occurs when the parameters are equal, (2) that the value of the maximum when $\rho = .25$ is .25(.25), which is in agreement with the known maximum of .25 for $\rho = 1$, (3) that the ratio k^*/m is essentially constant and equal to .7, k^* being the point at which the maximum effect occurs and m the mean life of the system.

A good approximation to the reliability difference can be obtained by the approximate method, ie, four times the estimate of the correlation coefficient times the product of the reliability and the unreliability of each component.

A procedure for obtaining a lower confidence limit for the reliability of a two component serial system is presented, however its usefulness is doubtful since it is in no way optimal. No better technique could be found although it is believed continued research in this area could be very fruitful.

TABLE 4.01

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .995 P2= .999 S= .996

K	RHO=.05	.10	.15	.20	.25
25.	.000501	.001001	.001502	.002003	.002503
50.	.001602	.003203	.004805	.006407	.008008
75.	.002886	.005772	.008658	.011544	.014430
100.	.004114	.008229	.012343	.016458	.020572
125.	.005162	.010325	.015487	.020649	.025812
150.	.005977	.011955	.017932	.023909	.029887
175.	.006550	.013101	.019651	.026202	.032752
200.	.006898	.013795	.020693	.027590	.034488
225.	.007047	.014094	.021141	.028188	.035235
250.	.007032	.014064	.021096	.028128	.035161
275.	.006886	.013772	.020659	.027545	.034431
300.	.006641	.013281	.019922	.026563	.033203
325.	.006323	.012646	.018969	.025292	.031615
350.	.005957	.011913	.017870	.023826	.029783
375.	.005561	.011122	.016682	.022243	.027804
400.	.005151	.010303	.015454	.020605	.025756
425.	.004740	.009480	.014220	.018960	.023701
450.	.004336	.008673	.013009	.017346	.021682
475.	.003947	.007894	.011841	.015788	.019735
500.	.003576	.007153	.010729	.014305	.017882
525.	.003228	.006455	.009683	.012911	.016138
550.	.002903	.005806	.008708	.011611	.014514
575.	.002602	.005205	.007807	.010409	.013012
600.	.002326	.004653	.006979	.009306	.011632
625.	.002074	.004149	.006223	.008298	.010372
650.	.001845	.003691	.005536	.007382	.009227
675.	.001638	.003277	.004915	.006554	.008192
700.	.001452	.002904	.004355	.005807	.007259
725.	.001284	.002568	.003853	.005137	.006421
750.	.001134	.002268	.003403	.004537	.005671
775.	.001000	.002001	.003001	.004001	.005001
800.	.000881	.001762	.002643	.003524	.004405
825.	.000775	.001550	.002325	.003100	.003875
850.	.000681	.001362	.002043	.002724	.003405
875.	.000598	.001195	.001793	.002391	.002989
900.	.000524	.001048	.001572	.002097	.002621
925.	.000459	.000918	.001378	.001837	.002296
950.	.000402	.000804	.001206	.001608	.002010
975.	.000352	.000703	.001055	.001407	.001758
1000.	.000307	.000615	.000922	.001229	.001537

TABLE 4.02

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .996 P2= .999 S= .997

K	RHO=.05	.10	.15	.20	.25
25.	.000416	.000831	.001247	.001662	.002078
50.	.001380	.002759	.004139	.005518	.006898
75.	.002578	.005157	.007735	.010313	.012891
100.	.003811	.007621	.011432	.015242	.019053
125.	.004954	.009908	.014862	.019817	.024771
150.	.005941	.011882	.017824	.023765	.029706
175.	.006740	.013481	.020221	.026962	.033702
200.	.007345	.014689	.022034	.029379	.036724
225.	.007762	.015523	.023285	.031047	.038808
250.	.008008	.016016	.024024	.032032	.040039
275.	.008104	.016209	.024313	.032417	.040522
300.	.008074	.016148	.024221	.032295	.040369
325.	.007939	.015877	.023816	.031754	.039693
350.	.007720	.015440	.023160	.030879	.038599
375.	.007437	.014874	.022310	.029747	.037184
400.	.007106	.014213	.021319	.028425	.035531
425.	.006743	.013486	.020228	.026971	.033714
450.	.006359	.012717	.019076	.025435	.031794
475.	.005964	.011928	.017892	.023857	.029821
500.	.005567	.011135	.016702	.022269	.027836
525.	.005175	.010349	.015524	.020699	.025874
550.	.004792	.009583	.014375	.019166	.023958
575.	.004422	.008843	.013265	.017686	.022108
600.	.004068	.008135	.012203	.016271	.020339
625.	.003732	.007463	.011195	.014927	.018659
650.	.003415	.006830	.010245	.013659	.017074
675.	.003118	.006235	.009353	.012471	.015589
700.	.002841	.005681	.008522	.011362	.014203
725.	.002583	.005166	.007749	.010332	.012915
750.	.002345	.004689	.007034	.009379	.011723
775.	.002125	.004250	.006375	.008500	.010625
800.	.001923	.003846	.005769	.007692	.009615
825.	.001738	.003476	.005213	.006951	.008689
850.	.001568	.003137	.004705	.006274	.007842
875.	.001414	.002828	.004242	.005656	.007070
900.	.001273	.002547	.003820	.005093	.006367
925.	.001145	.002291	.003436	.004582	.005727
950.	.001030	.002059	.003089	.004118	.005148
975.	.000925	.001849	.002774	.003698	.004623
1000.	.000830	.001659	.002489	.003318	.004148

TABLE 4.03

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .997 P2= .999 S= .998

K	RHO=.05	.10	.15	.20	.25
25.	.000323	.000647	.000970	.001294	.001617
50.	.001114	.002228	.003343	.004457	.005571
75.	.002160	.004320	.006480	.008641	.010801
100.	.003311	.006622	.009932	.013243	.016554
125.	.004462	.008924	.013386	.017849	.022311
150.	.005545	.011091	.016636	.022181	.027727
175.	.006517	.013034	.019552	.026069	.032586
200.	.007354	.014708	.022062	.029415	.036769
225.	.008045	.016089	.024134	.032179	.040223
250.	.008589	.017178	.025767	.034356	.042944
275.	.008992	.017984	.026976	.035968	.044960
300.	.009264	.018528	.027791	.037055	.046319
325.	.009416	.018833	.028249	.037665	.047082
350.	.009463	.018927	.028390	.037853	.047317
375.	.009418	.018837	.028255	.037673	.047092
400.	.009295	.018590	.027885	.037180	.046475
425.	.009106	.018213	.027319	.036425	.045532
450.	.008864	.017728	.026592	.035456	.044320
475.	.008579	.017158	.025738	.034317	.042896
500.	.008262	.016523	.024785	.033046	.041308
525.	.007920	.015839	.023759	.031679	.039598
550.	.007561	.015122	.022683	.030244	.037805
575.	.007192	.014384	.021577	.028769	.035961
600.	.006819	.013637	.020456	.027275	.034093
625.	.006445	.012890	.019335	.025780	.032225
650.	.006075	.012150	.018225	.024300	.030375
675.	.005712	.011424	.017136	.022848	.028560
700.	.005358	.010716	.016075	.021433	.026791
725.	.005016	.010031	.015047	.020063	.025078
750.	.004686	.009372	.014058	.018744	.023430
775.	.004370	.008740	.013110	.017480	.021850
800.	.004069	.008137	.012206	.016274	.020343
825.	.003782	.007564	.011346	.015128	.018910
850.	.003511	.007021	.010532	.014043	.017553
875.	.003254	.006509	.009763	.013018	.016272
900.	.003013	.006026	.009040	.012053	.015066
925.	.002787	.005573	.008360	.011146	.013933
950.	.002574	.005149	.007723	.010297	.012871
975.	.002376	.004751	.007127	.009503	.011879
1000.	.002190	.004381	.006571	.008762	.010952

TABLE 4.04

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .998 P2= .999 S= .999

K	RHO=.05	.10	.15	.20	.25
25.	.000224	.000447	.000671	.000895	.001119
50.	.000800	.001600	.002400	.003200	.004000
75.	.001609	.003219	.004828	.006437	.008047
100.	.002559	.005117	.007676	.010235	.012794
125.	.003576	.007153	.010729	.014306	.017882
150.	.004608	.009217	.013825	.018433	.023041
175.	.005614	.011228	.016842	.022456	.028070
200.	.006565	.013129	.019694	.026258	.032823
225.	.007440	.014880	.022320	.029760	.037200
250.	.008227	.016455	.024682	.032909	.041137
275.	.008919	.017839	.026758	.035677	.044596
300.	.009513	.019025	.028538	.038051	.047564
325.	.010008	.020016	.030023	.040031	.050039
350.	.010407	.020814	.031221	.041628	.052035
375.	.010715	.021430	.032145	.042860	.053574
400.	.010937	.021873	.032810	.043747	.054684
425.	.011079	.022158	.033237	.044316	.055395
450.	.011148	.022297	.033445	.044593	.055741
475.	.011152	.022303	.033455	.044607	.055759
500.	.011096	.022192	.033289	.044385	.055481
525.	.010989	.021977	.032966	.043955	.054943
550.	.010836	.021671	.032507	.043342	.054178
575.	.010643	.021286	.031929	.042572	.053215
600.	.010417	.020834	.031251	.041668	.052085
625.	.010163	.020326	.030489	.040652	.050815
650.	.009886	.019772	.029658	.039544	.049429
675.	.009590	.019180	.028771	.038361	.047951
700.	.009280	.018560	.027840	.037120	.046401
725.	.008959	.017919	.026878	.035837	.044797
750.	.008631	.017262	.025893	.034524	.043155
775.	.008298	.016597	.024895	.033194	.041492
800.	.007964	.015928	.023892	.031855	.039819
825.	.007630	.015259	.022889	.030519	.038148
850.	.007298	.014595	.021893	.029191	.036489
875.	.006970	.013940	.020909	.027879	.034849
900.	.006647	.013295	.019942	.026589	.033237
925.	.006331	.012663	.018994	.025326	.031657
950.	.006023	.012046	.018069	.024092	.030115
975.	.005723	.011446	.017169	.022892	.028615
1000.	.005432	.010864	.016296	.021729	.027161

TABLE 4.C5

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .999 P2= .999 S=1.000

K	RHO=.05	.10	.15	.20	.25
25.	.000116	.000232	.000348	.000464	.000580
50.	.000431	.000862	.001293	.001723	.002154
75.	.000900	.001799	.002699	.003598	.004498
100.	.001484	.002968	.004452	.005937	.007421
125.	.002152	.004305	.006457	.008609	.010762
150.	.002877	.005754	.008631	.011508	.014385
175.	.003635	.007270	.010906	.014541	.018176
200.	.004408	.008816	.013225	.017633	.022041
225.	.005180	.010361	.015541	.020722	.025902
250.	.005939	.011878	.017817	.023757	.029696
275.	.006674	.013348	.020022	.026697	.033371
300.	.007377	.014755	.022132	.029510	.036887
325.	.008043	.016086	.024128	.032171	.040214
350.	.008666	.017331	.025997	.034662	.043328
375.	.009243	.018485	.027728	.036970	.046213
400.	.009771	.019543	.029314	.039086	.048857
425.	.010251	.020502	.030754	.041005	.051256
450.	.010681	.021363	.032044	.042725	.053406
475.	.011062	.022124	.033186	.044248	.055310
500.	.011394	.022788	.034182	.045576	.056970
525.	.011679	.023357	.035036	.046714	.058393
550.	.011917	.023835	.035752	.047669	.059586
575.	.012112	.024224	.036336	.048448	.060560
600.	.012264	.024529	.036793	.049058	.061322
625.	.012377	.024754	.037131	.049509	.061886
650.	.012452	.024904	.037357	.049809	.062261
675.	.012492	.024984	.037476	.049968	.062460
700.	.012499	.024997	.037496	.049995	.062494
725.	.012475	.024950	.037425	.049900	.062374
750.	.012423	.024846	.037268	.049691	.062114
775.	.012345	.024689	.037034	.049379	.061723
800.	.012243	.024486	.036728	.048971	.061214
825.	.012119	.024238	.036358	.048477	.060596
850.	.011976	.023952	.035928	.047904	.059881
875.	.011815	.023631	.035446	.047262	.059077
900.	.011639	.023278	.034917	.046556	.058195
925.	.011449	.022897	.034346	.045795	.057244
950.	.011246	.022492	.033739	.044985	.056231
975.	.011033	.022066	.033099	.044132	.055165
1000.	.010811	.021622	.032433	.043243	.054054

TABLE 4.06

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .995 P2= .998 S= .997

K	RHO=.05	.10	.15	.20	.25
25.	.000965	.001930	.002895	.003860	.004825
50.	.002974	.005948	.008922	.011896	.014870
75.	.005163	.010326	.015490	.020653	.025816
100.	.007093	.014187	.021280	.028374	.035467
125.	.008578	.017156	.025734	.034312	.042891
150.	.009574	.019149	.028723	.038298	.047872
175.	.010116	.020232	.030348	.040464	.050580
200.	.010272	.020543	.030815	.041086	.051358
225.	.010121	.020241	.030362	.040483	.050603
250.	.009741	.019483	.029224	.038965	.048707
275.	.009203	.018405	.027608	.036811	.046013
300.	.008563	.017125	.025688	.034251	.042813
325.	.007868	.015735	.023603	.031471	.039339
350.	.007154	.014307	.021461	.028614	.035768
375.	.006447	.012893	.019340	.025787	.032233
400.	.005766	.011531	.017297	.023062	.028828
425.	.005123	.010246	.015369	.020491	.025614
450.	.004526	.009052	.013578	.018104	.022630
475.	.003979	.007958	.011937	.015916	.019895
500.	.003483	.006966	.010449	.013932	.017414
525.	.003037	.006074	.009111	.012148	.015185
550.	.002639	.005279	.007918	.010557	.013196
575.	.002287	.004573	.006860	.009147	.011434
600.	.001976	.003952	.005928	.007904	.009880
625.	.001703	.003407	.005110	.006813	.008517
650.	.001465	.002930	.004395	.005861	.007326
675.	.001258	.002516	.003773	.005031	.006289
700.	.001078	.002156	.003234	.004312	.005390
725.	.000922	.001845	.002767	.003689	.004612
750.	.000788	.001576	.002364	.003152	.003940
775.	.000672	.001345	.002017	.002690	.003362
800.	.000573	.001146	.001719	.002292	.002865
825.	.000488	.000976	.001464	.001952	.002439
850.	.000415	.000830	.001245	.001660	.002075
875.	.000353	.000705	.001058	.001410	.001763
900.	.000299	.000599	.000898	.001197	.001497
925.	.000254	.000508	.000762	.001016	.001270
950.	.000215	.000431	.000646	.000861	.001076
975.	.000182	.000365	.000547	.000730	.000912
1000.	.000154	.000309	.000463	.000618	.000772

TABLE 4.07

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .996 P2= .998 S= .998

K	RHO=.05	.10	.15	.20	.25
25.	.000801	.001602	.002403	.003204	.004005
50.	.002562	.005123	.007685	.010246	.012808
75.	.004613	.009225	.013838	.018451	.023063
100.	.006570	.013139	.019709	.026278	.032848
125.	.008232	.016464	.024696	.032929	.041161
150.	.009517	.019033	.028550	.038066	.047583
175.	.010410	.020819	.031229	.041638	.052048
200.	.010937	.021875	.032812	.043749	.054687
225.	.011147	.022294	.033441	.044588	.055735
250.	.011093	.022186	.033279	.044372	.055465
275.	.010831	.021661	.032492	.043322	.054153
300.	.010411	.020821	.031232	.041642	.052053
325.	.009878	.019756	.029634	.039512	.049391
350.	.009271	.018543	.027814	.037085	.046357
375.	.008621	.017243	.025864	.034486	.043107
400.	.007954	.015907	.023861	.031815	.039769
425.	.007287	.014575	.021862	.029149	.036436
450.	.006637	.013273	.019910	.026547	.033184
475.	.006013	.012025	.018038	.024050	.030063
500.	.005422	.010844	.016265	.021687	.027109
525.	.004869	.009738	.014607	.019476	.024345
550.	.004357	.008713	.013070	.017426	.021783
575.	.003885	.007771	.011656	.015542	.019427
600.	.003455	.006910	.010365	.013820	.017275
625.	.003064	.006128	.009192	.012257	.015321
650.	.002711	.005422	.008133	.010844	.013555
675.	.002394	.004787	.007181	.009574	.011968
700.	.002109	.004218	.006327	.008436	.010545
725.	.001855	.003710	.005565	.007420	.009275
750.	.001629	.003258	.004887	.006516	.008145
775.	.001428	.002857	.004285	.005714	.007142
800.	.001251	.002502	.003753	.005004	.006254
825.	.001094	.002188	.003282	.004376	.005470
850.	.000956	.001912	.002867	.003823	.004779
875.	.000834	.001668	.002502	.003336	.004170
900.	.000727	.001454	.002182	.002909	.003636
925.	.000633	.001267	.001900	.002534	.003167
950.	.000551	.001103	.001654	.002206	.002757
975.	.000480	.000959	.001439	.001918	.002398
1000.	.000417	.000834	.001250	.001667	.002084

TABLE 4.08

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .997 P2= .998 S= .999

K	RHO=.05	.10	.15	.20	.25
25.	.000623	.001247	.001870	.002494	.003117
50.	.002069	.004138	.006207	.008275	.010344
75.	.003865	.007729	.011594	.015459	.019323
100.	.005708	.011416	.017124	.022832	.028540
125.	.007415	.014829	.022244	.029659	.037073
150.	.008882	.017765	.026647	.035530	.044412
175.	.010065	.020129	.030194	.040259	.050324
200.	.010951	.021902	.032853	.043804	.054755
225.	.011553	.023107	.034660	.046214	.057767
250.	.011898	.023796	.035694	.047592	.059490
275.	.012017	.024034	.036051	.048067	.060084
300.	.011945	.023890	.035835	.047780	.059725
325.	.011717	.023434	.035151	.046868	.058585
350.	.011365	.022730	.034095	.045461	.056826
375.	.010919	.021837	.032756	.043675	.054594
400.	.010404	.020807	.031211	.041614	.052018
425.	.009842	.019683	.029525	.039366	.049208
450.	.009252	.018503	.027755	.037007	.046258
475.	.008649	.017298	.025947	.034595	.043244
500.	.008046	.016091	.024137	.032183	.040228
525.	.007452	.014904	.022355	.029807	.037259
550.	.006875	.013749	.020624	.027499	.034374
575.	.006320	.012640	.018960	.025280	.031600
600.	.005792	.011583	.017375	.023166	.028958
625.	.005292	.010584	.015876	.021168	.026460
650.	.004823	.009646	.014469	.019292	.024115
675.	.004385	.008770	.013155	.017541	.021926
700.	.003978	.007957	.011935	.015914	.019892
725.	.003602	.007204	.010807	.014409	.018011
750.	.003256	.006511	.009767	.013023	.016278
775.	.002938	.005875	.008813	.011750	.014688
800.	.002647	.005293	.007940	.010586	.013233
825.	.002381	.004762	.007143	.009524	.011905
850.	.002139	.004279	.006418	.008557	.010696
875.	.001920	.003840	.005759	.007679	.009599
900.	.001721	.003442	.005163	.006884	.008605
925.	.001541	.003082	.004623	.006164	.007705
950.	.001379	.002757	.004136	.005515	.006893
975.	.001232	.002465	.003697	.004929	.006162
1000.	.001101	.002201	.003302	.004402	.005503

TABLE 4.09

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .998 P2= .998 S=1.000

K	RHO=.05	.10	.15	.20	.25
25.	.000431	.000862	.001294	.001725	.002156
50.	.001485	.002971	.004456	.005942	.007427
75.	.002879	.005758	.008638	.011517	.014396
100.	.004411	.008823	.013234	.017645	.022057
125.	.005943	.011886	.017829	.023772	.029714
150.	.007382	.014763	.022145	.029526	.036908
175.	.008670	.017340	.026009	.034679	.043349
200.	.009775	.019551	.029326	.039102	.048877
225.	.010685	.021370	.032055	.042740	.053425
250.	.011397	.022794	.034191	.045588	.056985
275.	.011920	.023839	.035759	.047679	.059598
300.	.012266	.024532	.036798	.049064	.061330
325.	.012453	.024906	.037359	.049812	.062265
350.	.012499	.024997	.037496	.049994	.062493
375.	.012422	.024844	.037265	.049687	.062109
400.	.012241	.024482	.036723	.048964	.061205
425.	.011974	.023947	.035921	.047894	.059868
450.	.011636	.023271	.034907	.046543	.058179
475.	.011242	.022484	.033727	.044969	.056211
500.	.010806	.021613	.032419	.043225	.054032
525.	.010340	.020679	.031019	.041358	.051698
550.	.009852	.019704	.029556	.039408	.049260
575.	.009352	.018705	.028057	.037409	.046762
600.	.008848	.017696	.026544	.035392	.044240
625.	.008345	.016690	.025035	.033380	.041725
650.	.007848	.015697	.023545	.031394	.039242
675.	.007363	.014725	.022088	.029450	.036813
700.	.006890	.013781	.020671	.027561	.034452
725.	.006434	.012869	.019303	.025738	.032172
750.	.005997	.011993	.017990	.023987	.029984
775.	.005578	.011157	.016735	.022314	.027892
800.	.005180	.010361	.015541	.020722	.025902
825.	.004803	.009606	.014410	.019213	.024016
850.	.004447	.008894	.013341	.017788	.022235
875.	.004111	.008223	.012334	.016446	.020557
900.	.003796	.007593	.011389	.015186	.018982
925.	.003501	.007003	.010504	.014006	.017507
950.	.003226	.006451	.009677	.012903	.016129
975.	.002969	.005937	.008906	.011875	.014843
1000.	.002729	.005459	.008188	.010918	.013647

TABLE 4.1C

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .995 P2= .997 S= .998

K	RHO=.05	.10	.15	.20	.25
25.	.001395	.002790	.C04185	.CC5580	.006975
50.	.004142	.008284	.C12426	.016568	.02C710
75.	.006930	.013861	.020791	.C27722	.034652
100.	.009178	.018357	.C27535	.036714	.045892
125.	.010702	.021405	.032107	.C42810	.053512
150.	.011521	.023043	.C34564	.046085	.0576C6
175.	.011744	.023487	.C35231	.046975	.058718
200.	.0115C7	.023013	.034520	.C46026	.057533
225.	.010943	.021887	.032830	.C43773	.054717
250.	.010169	.020339	.C3C508	.040678	.050847
275.	.009278	.018555	.027833	.037111	.046389
300.	.008339	.016677	.C25016	.033354	.041693
325.	.007403	.014806	.C22208	.029611	.037014
350.	.006505	.01301C	.019515	.026019	.032524
375.	.005667	.011333	.C17000	.022666	.028333
400.	.004900	.009800	.014700	.019601	.C245C1
425.	.004211	.C08421	.C12632	.016843	.021054
450.	.003599	.007197	.C10796	.014395	.017993
475.	.003061	.006122	.009183	.012244	.0153C5
500.	.002593	.005186	.CC7779	.010373	.012966
525.	.002189	.004378	.C06566	.008755	.01C944
550.	.001842	.003683	.005525	.C07367	.009208
575.	.001545	.003091	.004636	.006181	.007726
600.	.001293	.002587	.C03880	.005174	.006467
625.	.001080	.002160	.003241	.CC4321	.0054C1
650.	.000900	.001801	.0C2701	.0036C1	.004502
675.	.000749	.001498	.002247	.C02997	.003746
700.	.000622	.001245	.001867	.C02490	.003112
725.	.000516	.001033	.CC1549	.002065	.002582
750.	.000428	.000856	.CC1284	.001711	.002139
775.	.000354	.000708	.001062	.C01416	.001770
800.	.000293	.00C586	.CC0878	.001171	.C01464
825.	.000242	.000484	.C00726	.C00967	.0012C9
850.	.000200	.000399	.C00599	.000798	.00C998
875.	.000165	.000329	.CC0494	.000659	.C00823
900.	.000136	.000271	.CC0407	.000543	.000678
925.	.000112	.000224	.000335	.C00447	.CC0559
950.	.000092	.000184	.000276	.C00368	.0CC460
975.	.000076	.000151	.0CC227	.000303	.000379
1000.	.000062	.000125	.C00187	.C00249	.000311

TABLE 4.11

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .996　　　P2= .997　　　S= .999

K	RHO=.05	.10	.15	.20	.25
25.	.001158	.002316	.003474	.004632	.005790
50.	.003568	.007136	.010703	.014271	.017839
75.	.006191	.012383	.018574	.024766	.030957
100.	.008501	.017001	.025502	.034002	.042503
125.	.010271	.020542	.030812	.041083	.051354
150.	.011452	.022903	.034355	.045807	.057258
175.	.012084	.024169	.036253	.048337	.060422
200.	.012252	.024505	.036757	.049010	.061262
225.	.012053	.024106	.036159	.048212	.060265
250.	.011581	.023161	.034742	.046322	.057903
275.	.010919	.021838	.032757	.043676	.054595
300.	.010138	.020276	.030414	.040553	.050691
325.	.009294	.018589	.027883	.037177	.046472
350.	.008431	.016861	.025292	.033722	.042153
375.	.007578	.015157	.022735	.030313	.037891
400.	.006760	.013520	.020279	.027039	.033799
425.	.005990	.011980	.017969	.023959	.029949
450.	.005277	.010554	.015831	.021108	.026385
475.	.004626	.009251	.013877	.018502	.023128
500.	.004037	.008073	.012110	.016147	.020184
525.	.003509	.007018	.010527	.014037	.017546
550.	.003040	.006080	.009120	.012160	.015200
575.	.002626	.005251	.007877	.010502	.013128
600.	.002262	.004523	.006785	.009046	.011308
625.	.001943	.003886	.005830	.007773	.009716
650.	.001666	.003332	.004998	.006664	.008330
675.	.001426	.002851	.004277	.005702	.007128
700.	.001218	.002435	.003653	.004871	.006089
725.	.001039	.002077	.003116	.004154	.005193
750.	.000884	.001769	.002653	.003538	.004422
775.	.000752	.001504	.002257	.003009	.003761
800.	.000639	.001278	.001917	.002556	.003195
825.	.000542	.001085	.001627	.002169	.002712
850.	.000460	.000920	.001379	.001839	.002299
875.	.000389	.000779	.001168	.001558	.001947
900.	.000330	.000659	.000989	.001319	.001648
925.	.000279	.000558	.000836	.001115	.001394
950.	.000236	.000471	.000707	.000943	.001178
975.	.000199	.000398	.000597	.000796	.000995
1000.	.000168	.000336	.000504	.000672	.000840

TABLE 4.12

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .997 P2= .997 S=1.000

K	RHO=.05	.10	.15	.20	.25
25.	.000901	.001802	.002703	.003605	.004506
50.	.002881	.005763	.008644	.011526	.014407
75.	.005187	.010375	.015562	.020750	.025937
100.	.007386	.014771	.022157	.029543	.036928
125.	.009251	.018502	.027753	.037003	.046254
150.	.010689	.021377	.032066	.042754	.053443
175.	.011684	.023368	.035052	.046736	.058420
200.	.012268	.024535	.036803	.049071	.061338
225.	.012493	.024985	.037478	.049970	.062463
250.	.012421	.024842	.037262	.049683	.062104
275.	.012115	.024230	.036344	.048459	.060574
300.	.011632	.023265	.034897	.046530	.058162
325.	.011025	.022049	.033074	.044098	.055123
350.	.010334	.020669	.031003	.041338	.051672
375.	.009598	.019195	.028793	.038390	.047988
400.	.008842	.017684	.026526	.035368	.044209
425.	.008089	.016179	.024268	.032357	.040447
450.	.007356	.014712	.022068	.029424	.036780
475.	.006654	.013307	.019961	.026615	.033269
500.	.005990	.011981	.017971	.023961	.029951
525.	.005371	.010741	.016112	.021482	.026853
550.	.004797	.009594	.014392	.019189	.023986
575.	.004271	.008542	.012812	.017083	.021354
600.	.003791	.007582	.011373	.015164	.018955
625.	.003356	.006712	.010068	.013424	.016780
650.	.002964	.005928	.008892	.011855	.014819
675.	.002612	.005224	.007835	.010447	.013059
700.	.002297	.004594	.006891	.009188	.011485
725.	.002017	.004033	.006050	.008067	.010083
750.	.001768	.003535	.005303	.007070	.008838
775.	.001547	.003094	.004641	.006188	.007735
800.	.001352	.002704	.004056	.005408	.006760
825.	.001180	.002361	.003541	.004721	.005901
850.	.001029	.002058	.003087	.004116	.005146
875.	.000896	.001793	.002689	.003586	.004482
900.	.000780	.001560	.002340	.003120	.003900
925.	.000678	.001357	.002035	.002713	.003391
950.	.000589	.001179	.001768	.002357	.002946
975.	.000512	.001023	.001535	.002046	.002558
1000.	.000444	.000888	.001331	.001775	.002219

TABLE 4.13

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .995 P2= .996 S= .999

K	RHO=.05	.10	.15	.20	.25
25.	.001792	.003585	.005377	.007170	.008962
50.	.005129	.010257	.015386	.020515	.025643
75.	.008272	.016544	.024816	.033087	.041359
100.	.010564	.021128	.031692	.042256	.052819
125.	.011882	.023765	.035647	.047530	.059412
150.	.012344	.024688	.037032	.049375	.061719
175.	.012146	.024292	.036438	.048584	.060730
200.	.011492	.022985	.034477	.045969	.057462
225.	.010558	.021117	.031675	.042234	.052792
250.	.009482	.018963	.028445	.037926	.047408
275.	.008362	.016724	.025086	.033448	.041809
300.	.007267	.014535	.021802	.029070	.036337
325.	.006241	.012482	.018723	.024964	.031205
350.	.005306	.010613	.015919	.021226	.026532
375.	.004474	.008949	.013423	.017898	.022372
400.	.003746	.007493	.011239	.014985	.018731
425.	.003118	.006236	.009354	.012471	.015589
450.	.002582	.005163	.007745	.010326	.012908
475.	.002128	.004256	.006384	.008512	.010640
500.	.001747	.003495	.005242	.006990	.008737
525.	.001430	.002860	.004291	.005721	.007151
550.	.001167	.002334	.003501	.004668	.005836
575.	.000950	.001900	.002850	.003800	.004750
600.	.000772	.001543	.002315	.003086	.003858
625.	.000625	.001251	.001876	.002502	.003127
650.	.000506	.001012	.001518	.002024	.002530
675.	.000409	.000818	.001227	.001636	.002045
700.	.000330	.000660	.000990	.001320	.001650
725.	.000266	.000532	.000798	.001064	.001330
750.	.000214	.000428	.000642	.000856	.001070
775.	.000172	.000344	.000517	.000689	.000861
800.	.000138	.000277	.000415	.000554	.000692
825.	.000111	.000222	.000333	.000445	.000556
850.	.000089	.000178	.000268	.000357	.000446
875.	.000072	.000143	.000215	.000286	.000358
900.	.000057	.000115	.000172	.000229	.000287
925.	.000046	.000092	.000138	.000184	.000230
950.	.000037	.000074	.000110	.000147	.000184
975.	.000029	.000059	.000088	.000118	.000147
1000.	.000024	.000047	.000071	.000094	.000118

TABLE 4.14

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .996 P2= .996 S=1.000

K	RHO=.05	.10	.15	.20	.25
25.	.001488	.002976	.004464	.005952	.007440
50.	.004418	.008835	.013253	.017670	.022088
75.	.007390	.014780	.022169	.029559	.036949
100.	.009784	.019567	.029351	.039134	.048918
125.	.011403	.022807	.034210	.045613	.057016
150.	.012269	.024539	.036808	.049077	.061346
175.	.012498	.024997	.037495	.049993	.062492
200.	.012237	.024475	.036712	.048949	.061187
225.	.011629	.023258	.034887	.046516	.058145
250.	.010797	.021594	.032392	.043189	.053986
275.	.009841	.019682	.029523	.039364	.049205
300.	.008836	.017672	.026507	.035343	.044179
325.	.007836	.015671	.023507	.031343	.039178
350.	.006877	.013755	.020632	.027509	.034387
375.	.005984	.011968	.017952	.023935	.029919
400.	.005168	.010336	.015504	.020672	.025840
425.	.004435	.008870	.013306	.017741	.022176
450.	.003785	.007571	.011356	.015142	.018927
475.	.003216	.006431	.009647	.012862	.016078
500.	.002720	.005441	.008161	.010881	.013601
525.	.002293	.004586	.006879	.009172	.011464
550.	.001927	.003853	.005780	.007706	.009633
575.	.001614	.003228	.004843	.006457	.008071
600.	.001349	.002698	.004047	.005396	.006746
625.	.001125	.002250	.003375	.004500	.005626
650.	.000936	.001873	.002809	.003746	.004682
675.	.000778	.001556	.002334	.003112	.003891
700.	.000646	.001291	.001937	.002582	.003228
725.	.000535	.001070	.001604	.002139	.002674
750.	.000443	.000885	.001328	.001770	.002213
775.	.000366	.000732	.001097	.001463	.001829
800.	.000302	.000604	.000906	.001208	.001510
825.	.000249	.000498	.000748	.000997	.001246
850.	.000205	.000411	.000616	.000822	.001027
875.	.000169	.000338	.000508	.000677	.000846
900.	.000139	.000279	.000418	.000557	.000697
925.	.000115	.000229	.000344	.000458	.000573
950.	.000094	.000188	.000283	.000377	.000471
975.	.000077	.000155	.000232	.000310	.000387
1000.	.000064	.000127	.000191	.000255	.000318

TABLE 4.15

TABLE OF GEOMETRIC RELIABILITY DIFFERENCES

P1= .995 P2= .995 S=1.000

K	RHO=.05	.10	.15	.20	.25
25.	.002159	.004319	.006478	.008638	.010797
50.	.005954	.011908	.017863	.023817	.029771
75.	.009259	.018518	.027778	.037037	.046296
100.	.011406	.022813	.034219	.045625	.057032
125.	.012382	.024764	.037146	.049527	.061909
150.	.012419	.024838	.037257	.049675	.062094
175.	.011804	.023607	.035411	.047214	.059018
200.	.010793	.021585	.032378	.043171	.053964
225.	.009586	.019173	.028759	.038345	.047931
250.	.008326	.016652	.024979	.033305	.041631
275.	.007105	.014210	.021315	.028420	.035525
300.	.005977	.011955	.017932	.023910	.029887
325.	.004971	.009942	.014912	.019883	.024854
350.	.004094	.008189	.012283	.016377	.020472
375.	.003346	.006691	.010037	.013383	.016728
400.	.002716	.005431	.008147	.010863	.013578
425.	.002192	.004384	.006575	.008767	.010959
450.	.001761	.003521	.005282	.007042	.008803
475.	.001408	.002817	.004225	.005633	.007041
500.	.001123	.002245	.003368	.004490	.005613
525.	.000892	.001784	.002676	.003568	.004460
550.	.000707	.001414	.002121	.002828	.003535
575.	.000559	.001118	.001677	.002237	.002796
600.	.000441	.000883	.001324	.001765	.002206
625.	.000348	.000695	.001043	.001391	.001738
650.	.000274	.000547	.000821	.001094	.001368
675.	.000215	.000430	.000645	.000860	.001074
700.	.000169	.000337	.000506	.000675	.000843
725.	.000132	.000264	.000397	.000529	.000661
750.	.000104	.000207	.000311	.000414	.000518
775.	.000081	.000162	.000243	.000324	.000405
800.	.000063	.000127	.000190	.000254	.000317
825.	.000050	.000099	.000149	.000198	.000248
850.	.000039	.000077	.000116	.000155	.000194
875.	.000030	.000060	.000091	.000121	.000151
900.	.000024	.000047	.000071	.000094	.000118
925.	.000018	.000037	.000055	.000074	.000092
950.	.000014	.000029	.000043	.000057	.000072
975.	.000011	.000022	.000034	.000045	.000056
1000.	.000009	.000017	.000026	.000035	.000044

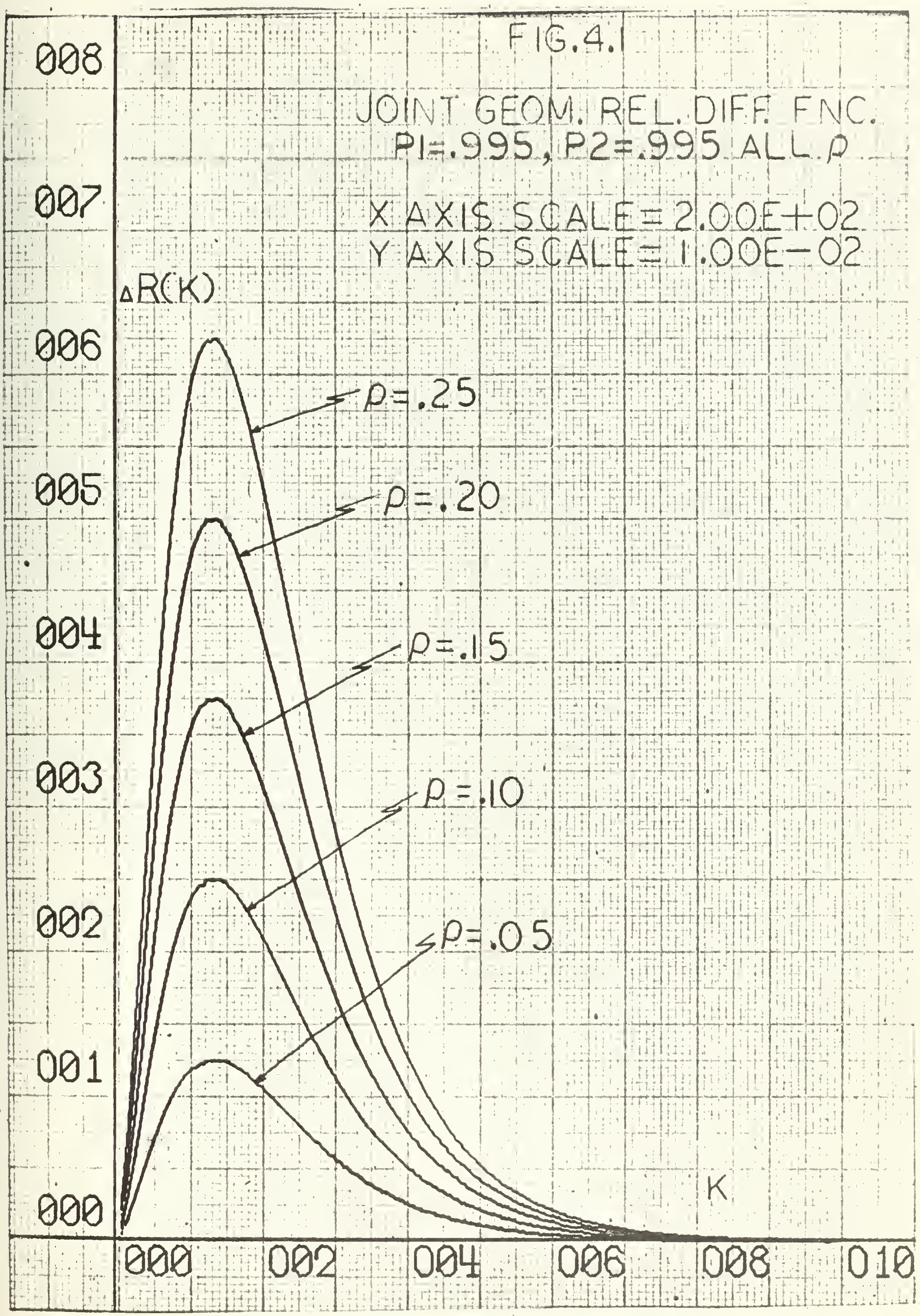
FIG.4.1
JOINT GEOM. REL. DIFF. FNC.
P1=.995, P2=.995 ALL ρ
X AXIS SCALE=2.00E+02
Y AXIS SCALE=1.00E-02
ΔR(K)
ρ=.25
ρ=.20
ρ=.15
ρ=.10
ρ=.05
K
008
007
006
005
004
003
002
001
000
000
002
004
006
008
010

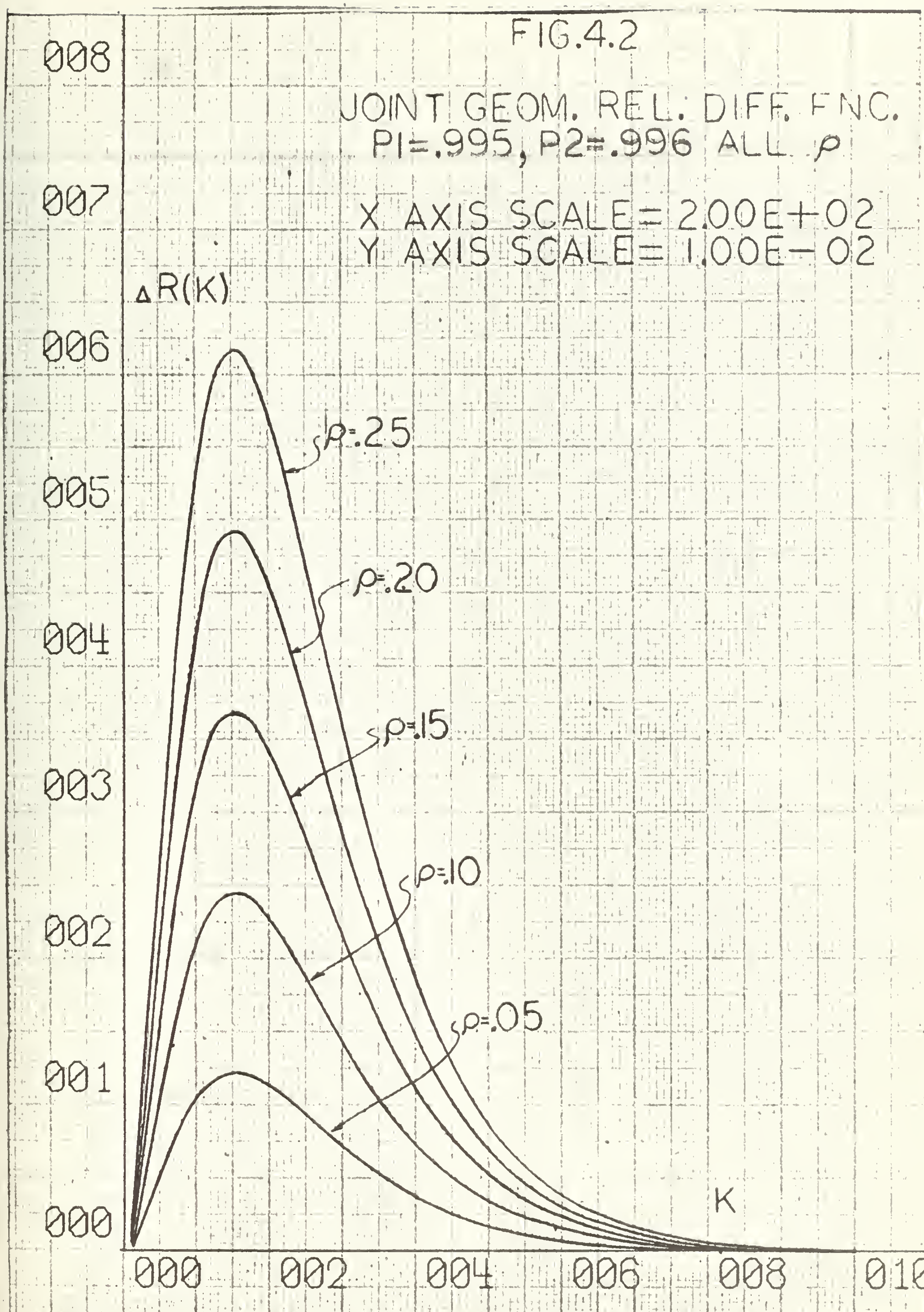
FIG.4.2
JOINT GEOM. REL. DIFF. FNC.
PI=.995, P2=.996 ALL ρ
X AXIS SCALE= 2.00E+02
Y AXIS SCALE= 1.00E-02
ΔR(K)
ρ=.25
ρ=.20
ρ=.15
ρ=.10
ρ=.05
K
008
007
006
005
004
003
002
001
000
000
002
004
006
008
010

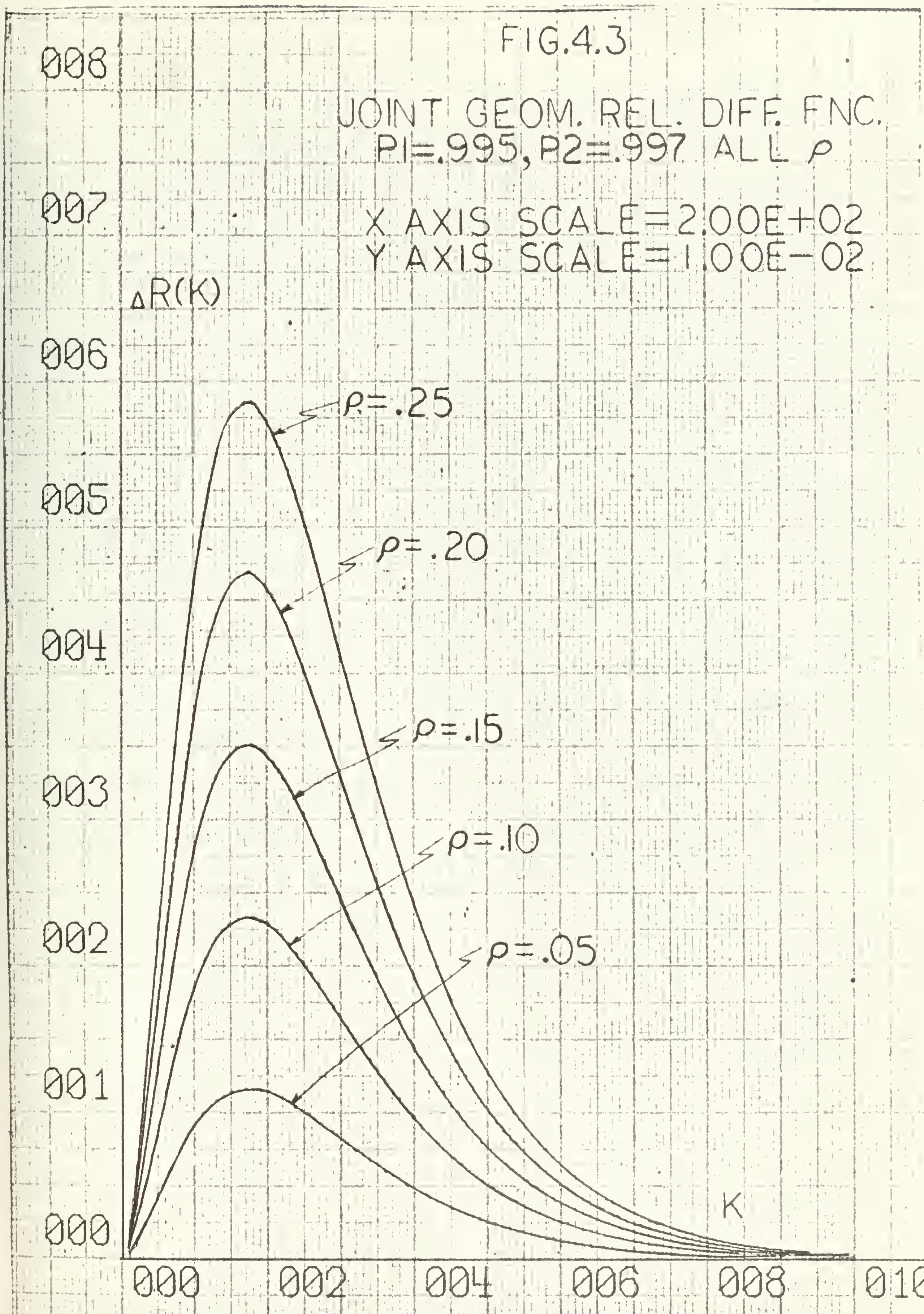
FIG.4.3
JOINT GEOM. REL. DIFF. FNC.
P1=.995, P2=.997 ALL ρ
X AXIS SCALE=2.00E+02
Y AXIS SCALE=1.00E-02
ΔR(K)
ρ=.25
ρ=.20
ρ=.15
ρ=.10
ρ=.05
K
008
007
006
005
004
003
002
001
000
000
002
004
006
008
010

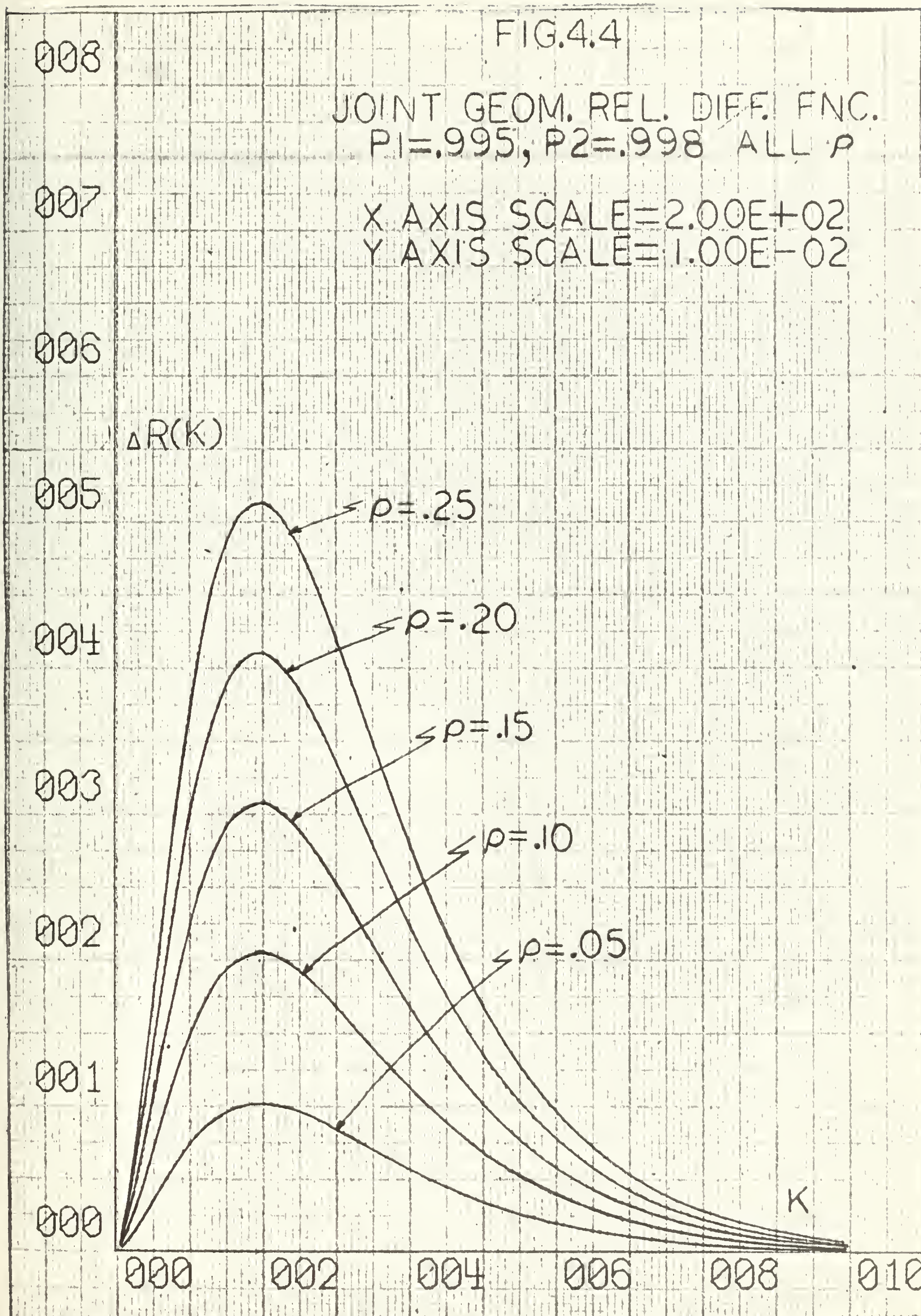

FIG.4.4
JOINT GEOM. REL. DIFF. FNC.
P1=.995, P2=.998 ALL ρ
X AXIS SCALE=2.00E+02
Y AXIS SCALE=1.00E-02
ΔR(K)
ρ=.25
ρ=.20
ρ=.15
ρ=.10
ρ=.05
K
008
007
006
005
004
003
002
001
000
000
002
004
006
008
010

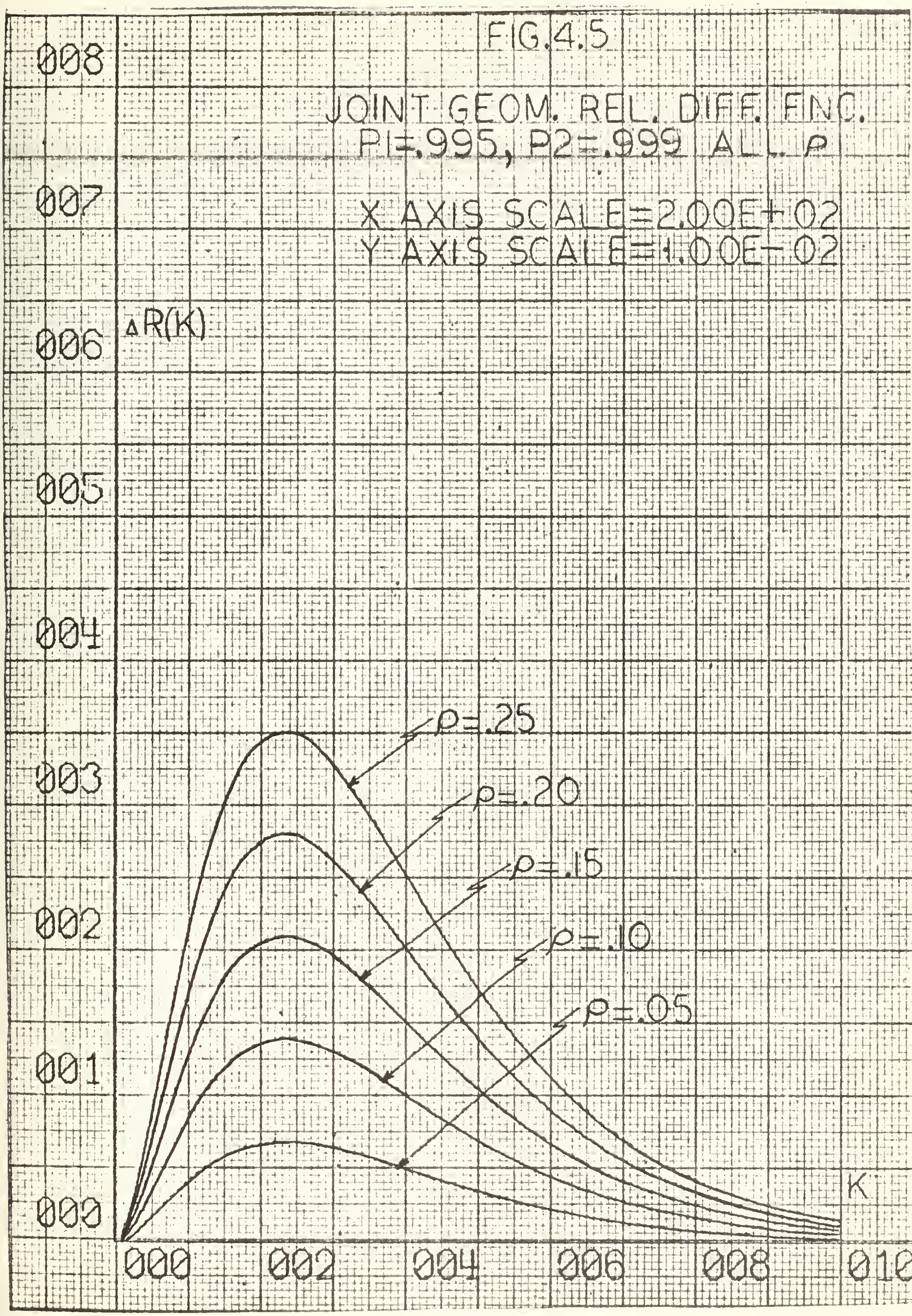
FIG.4.5
JOINT GEOM. REL. DIFF. FNC.
P1=.995, P2=.999 ALL ρ
X AXIS SCALE=2.00E+02
Y AXIS SCALE=1.00E-02
ΔR(K)
K
ρ=.25
ρ=.20
ρ=.15
ρ=.10
ρ=.05
000
001
002
003
004
005
006
007
008
010

Section 5

COMPOSITE FUNCTION

5.1 Derivation

As an extension of the two cases just considered we shall now investigate a system wherein the components that go to make up the system are continuous and discrete in nature. A simple example could be the operating life of an aircraft jet engine wherein the operating life depends on the number of starts as well as the total running time it accumulates. We may consider the starting function as a discrete random variable, specifically the geometric distribution, and the running time as a continuous random variable, specifically the exponential distribution.

With marginal density functions $f_X(x) = \frac{1}{a} \exp(-x/a)$ and $f_Y(y) = p^y (1-p)$ we again utilize the theory previously employed to generate a bivariate density function of the form

$$f_{XY}(x,y)=f_X(x)f_Y(y)\left[1+v(2F_X(x)-1)(2F_Y(y)-f_Y(y)-1)\right] \tag{5.1}$$

$$0\leq x\leq\infty \qquad y = 0, 1, 2, 3, \ldots$$

With the familiar restriction that $-1 \leq v \leq 1$ the function $f_{XY}(x,y)$ is shown in Appendix A.3 to satisfy all the requirements of a joint density function, ie, its sum over the range is unity, it is non-negative and its marginals are indeed the original density functions that went to make it up. The correlation coefficient is evaluated

and found to be

$$\rho = v\sqrt{p} \Big/ 2(1 + p) \tag{5.2}$$

where $-1 \leq v \leq 1$ and $0 \leq p \leq 1$.

It is again evident that the correlation is restricted to the range $-.25 \leq \rho \leq .25$. In the applications we shall consider, the value of p will generally be very high, between .990 and 1.000. For this range we see that the value of ρ is very nearly equal to v/4. This implies that for highly reliable items the correlation is completely specified by the constant v. Hence we shall consider the quantity $v = 4\,\rho$ for most computations. Tables 5.01 through 5.10 were computed using exact values.

5.2 System reliability

The system reliability is a function of the component reliabilities and can be expressed as

$$R(t,k) = P\left[X \geq t,\ Y \geq k \right] \tag{5.3}$$

The reliability function is evaluated in Appendix A.3 and the resultant expression for the system reliability is

$$R(t,k) = p^k \exp(-t/a)\left[1+v(1-\exp(-t/a)(1-p^k) \right] \tag{5.4}$$

The system reliability is seen to reduce to the product of the component reliabilities in the independent case wherein $v = 0$, as was to be expected.

To establish a quantitative measure of the effect of correlation on the system reliability a reliability difference function was defined as the difference between the system reliability when $\rho = 0$ and that when $\rho \neq 0$. This function is denoted by $\Delta R(t,k)$ and is expressed as

$$\Delta R(t,k) = v\, p^k \exp(-t/a)\left[1-\exp(-t/a)\right](1-p^k) \tag{5.5}$$

The reliability difference is seen to be a linear function of the correlation. This function has been extensively tabled in terms of the ratio of the total life to the mean life of the components. These are denoted by t/a and k/m in Tables 5.01 to 5.10. Further, the difference function is plotted in Figs. 5.1 and 5.2 and is seen to vary with k/m and is a maximum of .0625 at $k/m \approx .693$ for $\rho = .25$ and $t/a = .7$. This is in excellent agreement with previous results for the exponential and geometric cases.

5.3 Confidence Limits

The subject of deriving confidence limits for the reliability function defined above was considered beyond the scope of this thesis and was omitted. However, it is evident that this problem is of great importance and could well be the subject of a separate investigation.

5.4 Approximating the Effect

It is interesting to note that in the event that p is close to one the reliability difference function can be approximated very closely by the product of the component reliabilities and their unreliabilities times 4 ρ. That is, for $p \simeq 1$:

$$\Delta R(t,k) \doteq 4\rho \exp(-t/a) \left[1-\exp(-t/a)\right] p^k (1-p^k). \tag{5.6}$$

5.5 Summary

The bivariate density function derived from geometric and exponential marginals gives results consistent with those found in the previous two sections. Significantly, maximum effects of correlation on system reliability occurred for $k/m = .7$ in all cases and the extremum occurred when both k/m and t/a were about .7. Again this maximum effect was $\Delta R(t,m) = .0625$ for $\rho = .25$.

TABLE 5.01

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .100 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
5.0251	.00011	.00023	.00034	.00046	.00057
4.7739	.00015	.00029	.00044	.00058	.00073
4.5226	.00019	.00037	.00056	.00075	.00094
4.2714	.00024	.00048	.00072	.00096	.00120
4.0201	.00031	.00061	.00092	.00123	.00153
4.0161	.00031	.00061	.00092	.00123	.00154
3.8153	.00037	.00075	.00112	.00150	.00187
3.7688	.00039	.00078	.00118	.00157	.00196
3.6145	.00045	.00091	.00136	.00182	.00227
3.5176	.00050	.00100	.00150	.00200	.00250
3.4137	.00055	.00110	.00166	.00221	.00276
3.2663	.00064	.00127	.00191	.00255	.00318
3.2129	.00067	.00134	.00201	.00268	.00335
3.0151	.00081	.00162	.00243	.00324	.00404
3.0120	.00081	.00162	.00243	.00324	.00405
3.0090	.00081	.00162	.00243	.00324	.00406
2.8586	.00093	.00187	.00280	.00374	.00467
2.8112	.00098	.00196	.00294	.00391	.00489
2.7638	.00102	.00205	.00307	.00410	.00512
2.7081	.00108	.00215	.00323	.00430	.00538
2.6104	.00118	.00236	.00354	.00471	.00589
2.5577	.00124	.00247	.00371	.00494	.00618
2.5126	.00129	.00258	.00387	.00516	.00645
2.4096	.00141	.00283	.00424	.00566	.00707
2.4072	.00142	.00283	.00425	.00566	.00708
2.2613	.00162	.00323	.00485	.00646	.00808
2.2568	.00162	.00324	.00486	.00648	.00809
2.2088	.00169	.00338	.00507	.00676	.00845
2.1063	.00185	.00369	.00554	.00738	.00923
2.0101	.00201	.00401	.00602	.00803	.01003
2.0080	.00201	.00402	.00603	.00803	.01004
2.0040	.00201	.00402	.00604	.00805	.01006
1.9559	.00210	.00419	.00629	.00839	.01048
1.9038	.00219	.00437	.00656	.00875	.01094
1.8072	.00237	.00474	.00711	.00948	.01185
1.8054	.00237	.00474	.00711	.00948	.01185
1.8036	.00237	.00475	.00712	.00949	.01186
1.7588	.00246	.00493	.00739	.00986	.01232
1.7034	.00257	.00514	.00770	.01027	.01284
1.6550	.00267	.00533	.00800	.01067	.01334
1.6064	.00277	.00554	.00830	.01107	.01384
1.6032	.00277	.00554	.00831	.01109	.01386
1.5075	.00298	.00595	.00893	.01191	.01489
1.5045	.00298	.00596	.00894	.01192	.01490
1.5030	.00298	.00596	.00895	.01193	.01491
1.4056	.00319	.00639	.00958	.01277	.01597
1.4028	.00320	.00639	.00959	.01279	.01598
1.3541	.00330	.00661	.00991	.01321	.01651
1.3026	.00341	.00682	.01023	.01365	.01706
1.2563	.00351	.00703	.01054	.01406	.01757

TABLE 5.01

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .100 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
1.2048	.00362	.00724	.01086	.01448	.01810
1.2036	.00362	.00724	.01086	.01448	.01810
1.2024	.00362	.00724	.01087	.01449	.01811
1.1022	.00382	.00764	.01147	.01529	.01911
1.0532	.00391	.00783	.01174	.01566	.01957
1.0050	.00400	.00800	.01200	.01600	.02000
1.0040	.00400	.00800	.01200	.01601	.02001
1.0020	.00400	.00801	.01201	.01601	.02002
1.0010	.00400	.00801	.01201	.01602	.02002
.9510	.00408	.00817	.01225	.01633	.02042
.9027	.00415	.00831	.01246	.01661	.02077
.9018	.00415	.00831	.01246	.01662	.02077
.9009	.00415	.00831	.01246	.01662	.02077
.8509	.00421	.00843	.01264	.01686	.02107
.8032	.00426	.00852	.01278	.01704	.02130
.8016	.00426	.00852	.01278	.01704	.02130
.8008	.00426	.00852	.01278	.01704	.02130
.7538	.00429	.00858	.01287	.01717	.02146
.7523	.00429	.00858	.01288	.01717	.02146
.7508	.00429	.00858	.01288	.01717	.02146
.7014	.00431	.00861	.01292	.01722	.02153
.7007	.00431	.00861	.01292	.01722	.02153
.6507	.00430	.00859	.01289	.01719	.02149
.6024	.00427	.00853	.01280	.01706	.02133
.6018	.00427	.00853	.01280	.01706	.02133
.6012	.00426	.00853	.01279	.01706	.02132
.6006	.00426	.00853	.01279	.01706	.02132
.5506	.00420	.00841	.01261	.01682	.02102
.5025	.00411	.00823	.01234	.01645	.02056
.5010	.00411	.00822	.01233	.01644	.02056
.5005	.00411	.00822	.01233	.01644	.02055
.4514	.00398	.00796	.01194	.01592	.01991
.4505	.00398	.00796	.01194	.01592	.01990
.4016	.00381	.00762	.01143	.01524	.01904
.4008	.00381	.00761	.01142	.01523	.01904
.4004	.00381	.00761	.01142	.01523	.01903
.3504	.00358	.00717	.01075	.01434	.01792
.3009	.00331	.00662	.00993	.01324	.01655
.3006	.00331	.00662	.00993	.01323	.01654
.3003	.00331	.00662	.00992	.01323	.01654
.2513	.00297	.00594	.00891	.01189	.01486
.2503	.00297	.00594	.00890	.01187	.01484
.2008	.00256	.00512	.00768	.01024	.01280
.2004	.00256	.00512	.00767	.01023	.01279
.2002	.00256	.00511	.00767	.01023	.01278
.1505	.00207	.00413	.00620	.00827	.01034
.1502	.00207	.00413	.00620	.00826	.01033
.1002	.00148	.00297	.00445	.00594	.00742
.1001	.00148	.00297	.00445	.00593	.00742
.0000	.00000	.00000	.00000	.00000	.00000

TABLE 5.02

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .200   A=MEAN OF EXP DIST   M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
5.0251	.00020	.00039	.00059	.00078	.00098
4.7739	.00025	.00050	.00075	.00101	.00126
4.5226	.00032	.00064	.00097	.00129	.00161
4.2714	.00041	.00083	.00124	.00165	.00206
4.0201	.00053	.00106	.00159	.00211	.00264
4.0161	.00053	.00106	.00159	.00212	.00265
3.8153	.00064	.00129	.00193	.00258	.00322
3.7688	.00068	.00135	.00203	.00270	.00338
3.6145	.00078	.00157	.00235	.00313	.00392
3.5176	.00086	.00172	.00259	.00345	.00431
3.4137	.00095	.00190	.00285	.00380	.00476
3.2663	.00110	.00220	.00329	.00439	.00549
3.2129	.00115	.00231	.00346	.00461	.00577
3.0151	.00139	.00279	.00418	.00558	.00697
3.0120	.00140	.00279	.00419	.00558	.00698
3.0090	.00140	.00280	.00419	.00559	.00699
2.8586	.00161	.00322	.00483	.00644	.00806
2.8112	.00169	.00337	.00506	.00675	.00843
2.7638	.00176	.00353	.00529	.00706	.00882
2.7081	.00185	.00371	.00556	.00741	.00927
2.6104	.00203	.00406	.00609	.00812	.01016
2.5577	.00213	.00426	.00639	.00852	.01065
2.5126	.00222	.00445	.00667	.00889	.01112
2.4096	.00244	.00488	.00731	.00975	.01219
2.4072	.00244	.00488	.00732	.00976	.01220
2.2613	.00278	.00557	.00835	.01114	.01392
2.2568	.00279	.00558	.00837	.01116	.01395
2.2088	.00291	.00583	.00874	.01165	.01457
2.1063	.00318	.00636	.00954	.01272	.01591
2.0101	.00346	.00692	.01038	.01383	.01729
2.0080	.00346	.00692	.01038	.01385	.01731
2.0040	.00347	.00694	.01040	.01387	.01734
1.9559	.00361	.00723	.01084	.01445	.01807
1.9038	.00377	.00754	.01131	.01508	.01885
1.8072	.00408	.00817	.01225	.01633	.02042
1.8054	.00409	.00817	.01226	.01635	.02043
1.8036	.00409	.00818	.01227	.01636	.02045
1.7588	.00425	.00849	.01274	.01699	.02123
1.7034	.00443	.00885	.01328	.01770	.02213
1.6550	.00460	.00919	.01379	.01839	.02298
1.6064	.00477	.00954	.01431	.01909	.02386
1.6032	.00478	.00955	.01433	.01911	.02389
1.5075	.00513	.01026	.01539	.02053	.02566
1.5045	.00514	.01027	.01541	.02055	.02568
1.5030	.00514	.01028	.01542	.02056	.02570
1.4056	.00550	.01101	.01651	.02202	.02752
1.4028	.00551	.01102	.01653	.02204	.02755
1.3541	.00569	.01138	.01708	.02277	.02846
1.3026	.00588	.01176	.01764	.02352	.02940
1.2563	.00606	.01211	.01817	.02422	.03028

TABLE 5.02

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .200 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
1.2048	.00624	.01248	.01872	.02496	.03119
1.2036	.00624	.01248	.01872	.02496	.03120
1.2024	.00624	.01249	.01873	.02497	.03122
1.1022	.00659	.01318	.01976	.02635	.03294
1.0532	.00675	.01349	.02024	.02699	.03374
1.0050	.00690	.01379	.02069	.02758	.03448
1.0040	.00690	.01379	.02069	.02759	.03448
1.0020	.00690	.01380	.02070	.02760	.03450
1.0010	.00690	.01380	.02070	.02760	.03450
.9510	.00704	.01408	.02112	.02815	.03519
.9027	.00716	.01432	.02148	.02863	.03579
.9018	.00716	.01432	.02148	.02864	.03580
.9009	.00716	.01432	.02148	.02864	.03580
.8509	.00726	.01453	.02179	.02905	.03632
.8032	.00734	.01468	.02203	.02937	.03671
.8016	.00734	.01469	.02203	.02937	.03672
.8008	.00734	.01469	.02203	.02938	.03672
.7538	.00740	.01479	.02219	.02959	.03698
.7523	.00740	.01479	.02219	.02959	.03698
.7508	.00740	.01480	.02219	.02959	.03699
.7014	.00742	.01484	.02226	.02968	.03710
.7007	.00742	.01484	.02226	.02968	.03710
.6507	.00741	.01481	.02222	.02963	.03703
.6024	.00735	.01470	.02206	.02941	.03676
.6018	.00735	.01470	.02205	.02940	.03676
.6012	.00735	.01470	.02205	.02940	.03675
.6006	.00735	.01470	.02205	.02940	.03675
.5506	.00725	.01449	.02174	.02898	.03623
.5025	.00709	.01418	.02127	.02835	.03544
.5010	.00709	.01417	.02126	.02834	.03543
.5005	.00708	.01417	.02125	.02834	.03542
.4514	.00686	.01372	.02059	.02745	.03431
.4505	.00686	.01372	.02058	.02744	.03430
.4016	.00656	.01313	.01969	.02626	.03282
.4008	.00656	.01312	.01969	.02625	.03281
.4004	.00656	.01312	.01968	.02624	.03280
.3504	.00618	.01236	.01854	.02471	.03089
.3009	.00570	.01141	.01711	.02282	.02852
.3006	.00570	.01140	.01711	.02281	.02851
.3003	.00570	.01140	.01710	.02280	.02850
.2513	.00512	.01024	.01536	.02049	.02561
.2503	.00511	.01023	.01534	.02046	.02557
.2008	.00441	.00882	.01323	.01765	.02206
.2004	.00441	.00882	.01322	.01763	.02204
.2002	.00441	.00881	.01322	.01763	.02203
.1505	.00356	.00713	.01069	.01425	.01781
.1502	.00356	.00712	.01068	.01424	.01780
.1002	.00256	.00512	.00767	.01023	.01279
.1001	.00256	.00511	.00767	.01023	.01278
.0000	.00000	.00000	.00000	.00000	.00000

TABLE 5.03

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .300 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
5.0251	.00025	.00051	.00076	.00102	.00127
4.7739	.00033	.00065	.00098	.00130	.00163
4.5226	.00042	.00083	.00125	.00167	.00209
4.2714	.00053	.00107	.00160	.00214	.00267
4.0201	.00068	.00137	.00205	.00273	.00342
4.0161	.00069	.00137	.00206	.00274	.00343
3.8153	.00083	.00167	.00250	.00333	.00417
3.7688	.00087	.00175	.00262	.00350	.00437
3.6145	.00101	.00203	.00304	.00405	.00507
3.5176	.00112	.00223	.00335	.00446	.00558
3.4137	.00123	.00246	.00369	.00492	.00615
3.2663	.00142	.00284	.00426	.00568	.00710
3.2129	.00149	.00298	.00448	.00597	.00746
3.0151	.00180	.00361	.00541	.00722	.00902
3.0120	.00181	.00361	.00542	.00723	.00903
3.0090	.00181	.00362	.00543	.00724	.00904
2.8586	.00208	.00417	.00625	.00834	.01042
2.8112	.00218	.00436	.00655	.00873	.01091
2.7638	.00228	.00457	.00685	.00913	.01142
2.7081	.00240	.00480	.00719	.00959	.01199
2.6104	.00263	.00526	.00788	.01051	.01314
2.5577	.00275	.00551	.00826	.01102	.01377
2.5126	.00288	.00575	.00863	.01151	.01438
2.4096	.00315	.00631	.00946	.01262	.01577
2.4072	.00316	.00631	.00947	.01263	.01579
2.2613	.00360	.00721	.01081	.01441	.01801
2.2568	.00361	.00722	.01083	.01444	.01805
2.2088	.00377	.00754	.01131	.01508	.01884
2.1063	.00412	.00823	.01235	.01646	.02058
2.0101	.00447	.00895	.01342	.01790	.02237
2.0080	.00448	.00896	.01344	.01791	.02239
2.0040	.00449	.00897	.01346	.01794	.02243
1.9559	.00467	.00935	.01402	.01870	.02337
1.9038	.00488	.00975	.01463	.01951	.02438
1.8072	.00528	.01057	.01585	.02113	.02642
1.8054	.00529	.01057	.01586	.02115	.02643
1.8036	.00529	.01058	.01587	.02116	.02645
1.7588	.00549	.01099	.01648	.02198	.02747
1.7034	.00573	.01145	.01718	.02290	.02863
1.6550	.00595	.01189	.01784	.02379	.02974
1.6064	.00617	.01235	.01852	.02469	.03086
1.6032	.00618	.01236	.01854	.02472	.03090
1.5075	.00664	.01328	.01992	.02656	.03319
1.5045	.00665	.01329	.01994	.02658	.03323
1.5030	.00665	.01330	.01995	.02660	.03325
1.4056	.00712	.01424	.02136	.02848	.03561
1.4028	.00713	.01426	.02138	.02851	.03564
1.3541	.00736	.01473	.02209	.02946	.03682
1.3026	.00761	.01521	.02282	.03043	.03804
1.2563	.00784	.01567	.02351	.03134	.03918

TABLE 5.03

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .300 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
1.2048	.00807	.01614	.02421	.03229	.04036
1.2036	.00807	.01615	.02422	.03230	.04037
1.2024	.00808	.01615	.02423	.03231	.04039
1.1022	.00852	.01705	.02557	.03409	.04261
1.0532	.00873	.01746	.02619	.03492	.04365
1.0050	.00892	.01784	.02676	.03568	.04460
1.0040	.00892	.01785	.02677	.03569	.04461
1.0020	.00893	.01785	.02678	.03571	.04463
1.0010	.00893	.01786	.02678	.03571	.04464
.9510	.00911	.01821	.02732	.03642	.04553
.9027	.00926	.01852	.02778	.03704	.04631
.9018	.00926	.01852	.02779	.03705	.04631
.9009	.00926	.01853	.02779	.03706	.04632
.8509	.00940	.01879	.02819	.03759	.04698
.8032	.00950	.01900	.02850	.03800	.04749
.8016	.00950	.01900	.02850	.03800	.04750
.8008	.00950	.01900	.02850	.03800	.04751
.7538	.00957	.01914	.02871	.03828	.04785
.7523	.00957	.01914	.02871	.03828	.04785
.7508	.00957	.01914	.02871	.03828	.04785
.7014	.00960	.01920	.02880	.03840	.04800
.7007	.00960	.01920	.02880	.03840	.04800
.6507	.00958	.01916	.02875	.03833	.04791
.6024	.00951	.01902	.02853	.03805	.04756
.6018	.00951	.01902	.02853	.03804	.04755
.6012	.00951	.01902	.02853	.03804	.04755
.6006	.00951	.01902	.02853	.03804	.04755
.5506	.00937	.01875	.02812	.03750	.04687
.5025	.00917	.01834	.02751	.03668	.04585
.5010	.00917	.01833	.02750	.03667	.04584
.5005	.00917	.01833	.02750	.03666	.04583
.4514	.00888	.01776	.02663	.03551	.04439
.4505	.00887	.01775	.02662	.03550	.04437
.4016	.00849	.01699	.02548	.03397	.04247
.4008	.00849	.01698	.02547	.03396	.04245
.4004	.00849	.01698	.02546	.03395	.04244
.3504	.00799	.01599	.02398	.03197	.03997
.3009	.00738	.01476	.02214	.02952	.03690
.3006	.00738	.01475	.02213	.02951	.03689
.3003	.00738	.01475	.02213	.02950	.03688
.2513	.00663	.01325	.01988	.02650	.03313
.2503	.00662	.01323	.01985	.02647	.03309
.2008	.00571	.01141	.01712	.02283	.02854
.2004	.00570	.01141	.01711	.02281	.02852
.2002	.00570	.01140	.01710	.02280	.02851
.1505	.00461	.00922	.01383	.01844	.02305
.1502	.00461	.00921	.01382	.01842	.02303
.1002	.00331	.00662	.00993	.01324	.01655
.1001	.00331	.00662	.00992	.01323	.01654
.0000	.00000	.00000	.00000	.00000	.00000

TABLE 5.04

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .400   A=MEAN OF EXP DIST   M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
5.0251	.00029	.00058	.00088	.00117	.00146
4.7739	.00037	.00075	.00112	.00150	.00187
4.5226	.00048	.00096	.00144	.00192	.00240
4.2714	.00061	.00123	.00184	.00246	.00307
4.0201	.00079	.00157	.00236	.00315	.00393
4.0161	.00079	.00158	.00237	.00315	.00394
3.8153	.00096	.00192	.00288	.00384	.00480
3.7688	.00101	.00201	.00302	.00402	.00503
3.6145	.00117	.00233	.00350	.00467	.00583
3.5176	.00128	.00257	.00385	.00513	.00642
3.4137	.00142	.00283	.00425	.00567	.00708
3.2663	.00163	.00327	.00490	.00654	.00817
3.2129	.00172	.00344	.00515	.00687	.00859
3.0151	.00208	.00415	.00623	.00830	.01038
3.0120	.00208	.00416	.00624	.00832	.01040
3.0090	.00208	.00416	.00625	.00833	.01041
2.8586	.00240	.00480	.00720	.00960	.01200
2.8112	.00251	.00502	.00753	.01004	.01256
2.7638	.00263	.00526	.00788	.01051	.01314
2.7081	.00276	.00552	.00828	.01104	.01380
2.6104	.00302	.00605	.00907	.01210	.01512
2.5577	.00317	.00634	.00951	.01268	.01585
2.5126	.00331	.00662	.00993	.01324	.01656
2.4096	.00363	.00726	.01089	.01452	.01815
2.4072	.00363	.00727	.01090	.01454	.01817
2.2613	.00415	.00829	.01244	.01659	.02073
2.2568	.00416	.00831	.01247	.01662	.02078
2.2088	.00434	.00868	.01301	.01735	.02169
2.1063	.00474	.00947	.01421	.01895	.02368
2.0101	.00515	.01030	.01545	.02060	.02575
2.0080	.00515	.01031	.01546	.02062	.02577
2.0040	.00516	.01033	.01549	.02065	.02582
1.9559	.00538	.01076	.01614	.02152	.02690
1.9038	.00561	.01123	.01684	.02245	.02807
1.8072	.00608	.01216	.01824	.02432	.03040
1.8054	.00609	.01217	.01826	.02434	.03043
1.8036	.00609	.01218	.01827	.02436	.03045
1.7588	.00632	.01265	.01897	.02530	.03162
1.7034	.00659	.01318	.01977	.02636	.03295
1.6550	.00685	.01369	.02054	.02738	.03423
1.6064	.00710	.01421	.02131	.02842	.03552
1.6032	.00711	.01423	.02134	.02845	.03557
1.5075	.00764	.01528	.02292	.03056	.03820
1.5045	.00765	.01530	.02295	.03060	.03825
1.5030	.00765	.01531	.02296	.03061	.03827
1.4056	.00820	.01639	.02459	.03278	.04098
1.4028	.00820	.01641	.02461	.03281	.04102
1.3541	.00848	.01695	.02543	.03391	.04238
1.3026	.00876	.01751	.02627	.03502	.04378
1.2563	.00902	.01804	.02705	.03607	.04509

TABLE 5.04

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .400 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
1.2048	.00929	.01858	.02787	.03716	.04645
1.2036	.00929	.01859	.02788	.03717	.04647
1.2024	.00930	.01859	.02789	.03719	.04648
1.1022	.00981	.01962	.02943	.03924	.04905
1.0532	.01005	.02009	.03014	.04019	.05023
1.0050	.01027	.02053	.03080	.04107	.05134
1.0040	.01027	.02054	.03081	.04108	.05135
1.0020	.01027	.02055	.03082	.04109	.05137
1.0010	.01028	.02055	.03083	.04110	.05138
.9510	.01048	.02096	.03144	.04192	.05240
.9027	.01066	.02132	.03198	.04264	.05330
.9018	.01066	.02132	.03198	.04264	.05330
.9009	.01066	.02132	.03199	.04265	.05331
.8509	.01082	.02163	.03245	.04326	.05408
.8032	.01093	.02187	.03280	.04373	.05466
.8016	.01093	.02187	.03280	.04374	.05467
.8008	.01094	.02187	.03281	.04374	.05468
.7538	.01101	.02203	.03304	.04405	.05507
.7523	.01101	.02203	.03304	.04406	.05507
.7508	.01102	.02203	.03305	.04406	.05508
.7014	.01105	.02210	.03315	.04420	.05524
.7007	.01105	.02210	.03315	.04420	.05524
.6507	.01103	.02206	.03309	.04411	.05514
.6024	.01095	.02189	.03284	.04379	.05474
.6018	.01095	.02189	.03284	.04379	.05473
.6012	.01095	.02189	.03284	.04378	.05473
.6006	.01094	.02189	.03283	.04378	.05472
.5506	.01079	.02158	.03237	.04316	.05394
.5025	.01056	.02111	.03167	.04222	.05278
.5010	.01055	.02110	.03165	.04220	.05275
.5005	.01055	.02110	.03165	.04220	.05275
.4514	.01022	.02044	.03065	.04087	.05109
.4505	.01021	.02043	.03064	.04086	.05107
.4016	.00978	.01955	.02933	.03910	.04888
.4008	.00977	.01954	.02931	.03909	.04886
.4004	.00977	.01954	.02931	.03908	.04885
.3504	.00920	.01840	.02760	.03680	.04600
.3009	.00849	.01699	.02548	.03397	.04247
.3006	.00849	.01698	.02547	.03396	.04246
.3003	.00849	.01698	.02547	.03395	.04244
.2513	.00763	.01525	.02288	.03050	.03813
.2503	.00762	.01523	.02285	.03047	.03808
.2008	.00657	.01314	.01971	.02627	.03284
.2004	.00656	.01313	.01969	.02626	.03282
.2002	.00656	.01312	.01969	.02625	.03281
.1505	.00531	.01061	.01592	.02122	.02653
.1502	.00530	.01060	.01590	.02120	.02650
.1002	.00381	.00762	.01143	.01524	.01905
.1001	.00381	.00761	.01142	.01523	.01904
.0000	.00000	.00000	.00000	.00000	.00000

TABLE 5.05

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .500 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
5.0251	.00032	.00063	.00095	.00126	.00158
4.7739	.00040	.00081	.00121	.00162	.00202
4.5226	.00052	.00104	.00156	.00207	.00259
4.2714	.00066	.00133	.00199	.00266	.00332
4.0201	.00085	.00170	.00255	.00340	.00425
4.0161	.00085	.00170	.00255	.00341	.00426
3.8153	.00104	.00207	.00311	.00414	.00518
3.7688	.00109	.00217	.00326	.00434	.00543
3.6145	.00126	.00252	.00378	.00504	.00630
3.5176	.00139	.00277	.00416	.00554	.00693
3.4137	.00153	.00306	.00459	.00612	.00765
3.2663	.00177	.00353	.00530	.00706	.00883
3.2129	.00185	.00371	.00556	.00742	.00927
3.0151	.00224	.00448	.00673	.00897	.01121
3.0120	.00225	.00449	.00674	.00898	.01123
3.0090	.00225	.00450	.00675	.00899	.01124
2.8586	.00259	.00518	.00777	.01036	.01295
2.8112	.00271	.00542	.00814	.01085	.01356
2.7638	.00284	.00568	.00851	.01135	.01419
2.7081	.00298	.00596	.00894	.01192	.01490
2.6104	.00327	.00653	.00980	.01306	.01633
2.5577	.00342	.00685	.01027	.01370	.01712
2.5126	.00358	.00715	.01073	.01430	.01788
2.4096	.00392	.00784	.01176	.01568	.01960
2.4072	.00392	.00785	.01177	.01570	.01962
2.2613	.00448	.00896	.01343	.01791	.02239
2.2568	.00449	.00897	.01346	.01795	.02244
2.2088	.00468	.00937	.01405	.01874	.02342
2.1063	.00512	.01023	.01535	.02046	.02558
2.0101	.00556	.01112	.01669	.02225	.02781
2.0080	.00557	.01113	.01670	.02227	.02783
2.0040	.00558	.01115	.01673	.02230	.02788
1.9559	.00581	.01162	.01743	.02324	.02905
1.9038	.00606	.01212	.01819	.02425	.03031
1.8072	.00657	.01313	.01970	.02627	.03283
1.8054	.00657	.01314	.01971	.02629	.03286
1.8036	.00658	.01315	.01973	.02630	.03288
1.7588	.00683	.01366	.02049	.02732	.03415
1.7034	.00712	.01423	.02135	.02847	.03559
1.6550	.00739	.01478	.02218	.02957	.03696
1.6064	.00767	.01535	.02302	.03069	.03836
1.6032	.00768	.01536	.02305	.03073	.03841
1.5075	.00825	.01650	.02475	.03301	.04126
1.5045	.00826	.01652	.02478	.03304	.04130
1.5030	.00826	.01653	.02479	.03306	.04132
1.4056	.00885	.01770	.02655	.03540	.04425
1.4028	.00886	.01772	.02658	.03544	.04430
1.3541	.00915	.01831	.02746	.03662	.04577
1.3026	.00946	.01891	.02837	.03782	.04728
1.2563	.00974	.01948	.02922	.03895	.04869

TABLE 5.05

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .500 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
1.2048	.01003	.02006	.03010	.04013	.05016
1.2036	.01004	.02007	.03011	.04014	.05018
1.2024	.01004	.02008	.03012	.04016	.05020
1.1022	.01059	.02119	.03178	.04237	.05297
1.0532	.01085	.02170	.03255	.04340	.05425
1.0050	.01109	.02218	.03326	.04435	.05544
1.0040	.01109	.02218	.03327	.04436	.05545
1.0020	.01109	.02219	.03328	.04438	.05547
1.0010	.01110	.02219	.03329	.04439	.05549
.9510	.01132	.02264	.03395	.04527	.05659
.9027	.01151	.02302	.03453	.04604	.05756
.9018	.01151	.02303	.03454	.04605	.05756
.9009	.01151	.02303	.03454	.04606	.05757
.8509	.01168	.02336	.03504	.04672	.05840
.8032	.01181	.02361	.03542	.04723	.05903
.8016	.01181	.02362	.03542	.04723	.05904
.8008	.01181	.02362	.03543	.04724	.05905
.7538	.01189	.02379	.03568	.04758	.05947
.7523	.01189	.02379	.03568	.04758	.05947
.7508	.01190	.02379	.03569	.04758	.05948
.7014	.01193	.02386	.03580	.04773	.05966
.7007	.01193	.02386	.03580	.04773	.05966
.6507	.01191	.02382	.03573	.04764	.05955
.6024	.01182	.02364	.03547	.04729	.05911
.6018	.01182	.02364	.03546	.04728	.05911
.6012	.01182	.02364	.03546	.04728	.05910
.6006	.01182	.02364	.03546	.04728	.05910
.5506	.01165	.02330	.03495	.04660	.05826
.5025	.01140	.02280	.03420	.04559	.05699
.5010	.01139	.02279	.03418	.04558	.05697
.5005	.01139	.02278	.03418	.04557	.05696
.4514	.01103	.02207	.03310	.04414	.05517
.4505	.01103	.02206	.03309	.04412	.05515
.4016	.01056	.02111	.03167	.04223	.05278
.4008	.01055	.02110	.03166	.04221	.05276
.4004	.01055	.02110	.03165	.04220	.05275
.3504	.00994	.01987	.02981	.03974	.04968
.3009	.00917	.01834	.02752	.03669	.04586
.3006	.00917	.01834	.02751	.03668	.04585
.3003	.00917	.01833	.02750	.03667	.04584
.2513	.00824	.01647	.02471	.03294	.04118
.2503	.00823	.01645	.02468	.03290	.04113
.2008	.00709	.01419	.02128	.02837	.03547
.2004	.00709	.01418	.02127	.02835	.03544
.2002	.00709	.01417	.02126	.02834	.03543
.1505	.00573	.01146	.01719	.02292	.02865
.1502	.00572	.01145	.01717	.02290	.02862
.1002	.00411	.00823	.01234	.01645	.02057
.1001	.00411	.00822	.01233	.01645	.02056
.0000	.00000	.00000	.00000	.00000	.00000

TABLE 5.06

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .600   A=MEAN OF EXP DIST   M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
5.0251	.00033	.00065	.00098	.00131	.00164
4.7739	.00042	.00084	.00126	.00168	.00210
4.5226	.00054	.00108	.00161	.00215	.00269
4.2714	.00069	.00138	.00207	.00276	.00345
4.0201	.00088	.00176	.00265	.00353	.00441
4.0161	.00088	.00177	.00265	.00353	.00442
3.8153	.00108	.00215	.00323	.00430	.00538
3.7688	.00113	.00225	.00338	.00451	.00563
3.6145	.00131	.00261	.00392	.00523	.00653
3.5176	.00144	.00288	.00431	.00575	.00719
3.4137	.00159	.00317	.00476	.00635	.00794
3.2663	.00183	.00366	.00549	.00733	.00916
3.2129	.00192	.00385	.00577	.00770	.00962
3.0151	.00233	.00465	.00698	.00930	.01163
3.0120	.00233	.00466	.00699	.00932	.01165
3.0090	.00233	.00467	.00700	.00933	.01166
2.8586	.00269	.00538	.00806	.01075	.01344
2.8112	.00281	.00563	.00844	.01125	.01407
2.7638	.00294	.00589	.00883	.01178	.01472
2.7081	.00309	.00619	.00928	.01237	.01546
2.6104	.00339	.00678	.01017	.01356	.01694
2.5577	.00355	.00710	.01066	.01421	.01776
2.5126	.00371	.00742	.01113	.01484	.01855
2.4096	.00407	.00813	.01220	.01627	.02034
2.4072	.00407	.00814	.01222	.01629	.02036
2.2613	.00465	.00929	.01394	.01859	.02323
2.2568	.00466	.00931	.01397	.01862	.02328
2.2088	.00486	.00972	.01458	.01944	.02430
2.1063	.00531	.01061	.01592	.02123	.02654
2.0101	.00577	.01154	.01731	.02308	.02885
2.0080	.00578	.01155	.01733	.02310	.02888
2.0040	.00579	.01157	.01736	.02314	.02893
1.9559	.00603	.01206	.01809	.02411	.03014
1.9038	.00629	.01258	.01887	.02516	.03145
1.8072	.00681	.01363	.02044	.02725	.03407
1.8054	.00682	.01364	.02045	.02727	.03409
1.8036	.00682	.01365	.02047	.02729	.03412
1.7588	.00709	.01417	.02126	.02834	.03543
1.7034	.00738	.01477	.02215	.02954	.03692
1.6550	.00767	.01534	.02301	.03068	.03835
1.6064	.00796	.01592	.02388	.03184	.03980
1.6032	.00797	.01594	.02391	.03188	.03985
1.5075	.00856	.01712	.02568	.03425	.04281
1.5045	.00857	.01714	.02571	.03428	.04285
1.5030	.00858	.01715	.02573	.03430	.04288
1.4056	.00918	.01837	.02755	.03673	.04592
1.4028	.00919	.01838	.02758	.03677	.04596
1.3541	.00950	.01900	.02849	.03799	.04749
1.3026	.00981	.01962	.02943	.03924	.04905
1.2563	.01010	.02021	.03031	.04042	.05052

TABLE 5.06

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .600   A=MEAN OF EXP DIST   M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
1.2048	.01041	.02082	.03123	.04164	.05205
1.2036	.01041	.02083	.03124	.04165	.05206
1.2024	.01042	.02083	.03125	.04167	.05208
1.1022	.01099	.02198	.03297	.04397	.05496
1.0532	.01126	.02251	.03377	.04503	.05629
1.0050	.01150	.02301	.03451	.04602	.05752
1.0040	.01151	.02301	.03452	.04603	.05753
1.0020	.01151	.02302	.03453	.04605	.05756
1.0010	.01151	.02303	.03454	.04606	.05757
.9510	.01174	.02349	.03523	.04697	.05872
.9027	.01194	.02389	.03583	.04777	.05972
.9018	.01195	.02389	.03584	.04778	.05973
.9009	.01195	.02389	.03584	.04779	.05973
.8509	.01212	.02424	.03636	.04847	.06059
.8032	.01225	.02450	.03675	.04900	.06125
.8016	.01225	.02450	.03676	.04901	.06126
.8008	.01225	.02451	.03676	.04901	.06126
.7538	.01234	.02468	.03702	.04936	.06170
.7523	.01234	.02468	.03702	.04937	.06171
.7508	.01234	.02469	.03703	.04937	.06171
.7014	.01238	.02476	.03714	.04952	.06190
.7007	.01238	.02476	.03714	.04952	.06190
.6507	.01236	.02471	.03707	.04943	.06179
.6024	.01227	.02453	.03680	.04906	.06133
.6018	.01227	.02453	.03680	.04906	.06133
.6012	.01226	.02453	.03679	.04906	.06132
.6006	.01226	.02453	.03679	.04905	.06132
.5506	.01209	.02418	.03627	.04836	.06044
.5025	.01183	.02365	.03548	.04731	.05913
.5010	.01182	.02364	.03547	.04729	.05911
.5005	.01182	.02364	.03546	.04728	.05910
.4514	.01145	.02290	.03435	.04579	.05724
.4505	.01144	.02289	.03433	.04578	.05722
.4016	.01095	.02191	.03286	.04381	.05477
.4008	.01095	.02190	.03285	.04380	.05474
.4004	.01095	.02189	.03284	.04379	.05473
.3504	.01031	.02062	.03093	.04123	.05154
.3009	.00952	.01903	.02855	.03807	.04758
.3006	.00951	.01903	.02854	.03806	.04757
.3003	.00951	.01902	.02853	.03805	.04756
.2513	.00854	.01709	.02563	.03418	.04272
.2503	.00853	.01707	.02560	.03414	.04267
.2008	.00736	.01472	.02208	.02944	.03680
.2004	.00735	.01471	.02206	.02942	.03677
.2002	.00735	.01470	.02206	.02941	.03676
.1505	.00594	.01189	.01783	.02378	.02972
.1502	.00594	.01188	.01782	.02376	.02970
.1002	.00427	.00854	.01280	.01707	.02134
.1001	.00427	.00853	.01280	.01706	.02133
.0000	.00000	.00000	.00000	.00000	.00000

TABLE 5.07

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .700 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
5.0251	.00033	.00066	.00099	.00132	.00165
4.7739	.00042	.00085	.00127	.00170	.00212
4.5226	.00054	.00109	.00163	.00217	.00272
4.2714	.00070	.00139	.00209	.00278	.00348
4.0201	.00089	.00178	.00267	.00356	.00445
4.0161	.00089	.00178	.00268	.00357	.00446
3.8153	.00109	.00217	.00326	.00434	.00543
3.7688	.00114	.00228	.00341	.00455	.00569
3.6145	.00132	.00264	.00396	.00528	.00660
3.5176	.00145	.00290	.00436	.00581	.00726
3.4137	.00160	.00320	.00481	.00641	.00801
3.2663	.00185	.00370	.00555	.00740	.00924
3.2129	.00194	.00389	.00583	.00777	.00971
3.0151	.00235	.00470	.00705	.00939	.01174
3.0120	.00235	.00470	.00706	.00941	.01176
3.0090	.00236	.00471	.00707	.00942	.01178
2.8586	.00271	.00543	.00814	.01086	.01357
2.8112	.00284	.00568	.00852	.01136	.01420
2.7638	.00297	.00595	.00892	.01189	.01486
2.7081	.00312	.00625	.00937	.01249	.01561
2.6104	.00342	.00684	.01026	.01368	.01711
2.5577	.00359	.00717	.01076	.01435	.01793
2.5126	.00375	.00749	.01124	.01498	.01873
2.4096	.00411	.00821	.01232	.01643	.02053
2.4072	.00411	.00822	.01233	.01644	.02055
2.2613	.00469	.00938	.01407	.01876	.02345
2.2568	.00470	.00940	.01410	.01880	.02350
2.2088	.00491	.00981	.01472	.01963	.02454
2.1063	.00536	.01072	.01607	.02143	.02679
2.0101	.00583	.01165	.01748	.02330	.02913
2.0080	.00583	.01166	.01749	.02332	.02915
2.0040	.00584	.01168	.01752	.02336	.02920
1.9559	.00609	.01217	.01826	.02435	.03043
1.9038	.00635	.01270	.01905	.02540	.03175
1.8072	.00688	.01376	.02064	.02751	.03439
1.8054	.00688	.01377	.02065	.02753	.03442
1.8036	.00689	.01378	.02067	.02755	.03444
1.7588	.00715	.01431	.02146	.02861	.03577
1.7034	.00746	.01491	.02237	.02982	.03728
1.6550	.00774	.01549	.02323	.03097	.03872
1.6064	.00804	.01607	.02411	.03215	.04019
1.6032	.00805	.01609	.02414	.03219	.04023
1.5075	.00864	.01729	.02593	.03457	.04322
1.5045	.00865	.01731	.02596	.03461	.04326
1.5030	.00866	.01731	.02597	.03463	.04329
1.4056	.00927	.01854	.02781	.03709	.04636
1.4028	.00928	.01856	.02784	.03712	.04640
1.3541	.00959	.01918	.02877	.03835	.04794
1.3026	.00990	.01981	.02971	.03962	.04952
1.2563	.01020	.02040	.03060	.04081	.05101

TABLE 5.07

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .700 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
1.2048	.01051	.02102	.03153	.04204	.05254
1.2036	.01051	.02103	.03154	.04205	.05256
1.2024	.01052	.02103	.03155	.04206	.05258
1.1022	.01110	.02219	.03329	.04439	.05548
1.0532	.01137	.02273	.03410	.04546	.05683
1.0050	.01161	.02323	.03484	.04646	.05807
1.0040	.01162	.02323	.03485	.04647	.05808
1.0020	.01162	.02324	.03487	.04649	.05811
1.0010	.01162	.02325	.03487	.04650	.05812
.9510	.01186	.02371	.03557	.04742	.05928
.9027	.01206	.02412	.03617	.04823	.06029
.9018	.01206	.02412	.03618	.04824	.06030
.9009	.01206	.02412	.03618	.04825	.06031
.8509	.01223	.02447	.03670	.04894	.06117
.8032	.01237	.02473	.03710	.04947	.06184
.8016	.01237	.02474	.03711	.04948	.06185
.8008	.01237	.02474	.03711	.04948	.06185
.7538	.01246	.02492	.03738	.04984	.06229
.7523	.01246	.02492	.03738	.04984	.06230
.7508	.01246	.02492	.03738	.04984	.06230
.7014	.01250	.02500	.03750	.04999	.06249
.7007	.01250	.02500	.03750	.05000	.06249
.6507	.01248	.02495	.03743	.04990	.06238
.6024	.01238	.02477	.03715	.04953	.06192
.6018	.01238	.02477	.03715	.04953	.06191
.6012	.01238	.02476	.03715	.04953	.06191
.6006	.01238	.02476	.03714	.04952	.06191
.5506	.01220	.02441	.03661	.04882	.06102
.5025	.01194	.02388	.03582	.04776	.05970
.5010	.01194	.02387	.03581	.04774	.05968
.5005	.01193	.02387	.03580	.04773	.05967
.4514	.01156	.02312	.03467	.04623	.05779
.4505	.01155	.02311	.03466	.04622	.05777
.4016	.01106	.02212	.03317	.04423	.05529
.4008	.01105	.02211	.03316	.04421	.05527
.4004	.01105	.02210	.03315	.04421	.05526
.3504	.01041	.02081	.03122	.04163	.05204
.3009	.00961	.01922	.02882	.03843	.04804
.3006	.00961	.01921	.02882	.03842	.04803
.3003	.00960	.01921	.02881	.03841	.04801
.2513	.00863	.01725	.02588	.03451	.04313
.2503	.00862	.01723	.02585	.03446	.04308
.2008	.00743	.01486	.02229	.02972	.03715
.2004	.00743	.01485	.02228	.02970	.03713
.2002	.00742	.01485	.02227	.02969	.03711
.1505	.00600	.01200	.01800	.02400	.03001
.1502	.00600	.01199	.01799	.02399	.02998
.1002	.00431	.00862	.01293	.01724	.02154
.1001	.00431	.00861	.01292	.01723	.02153
.0000	.00000	.00000	.00000	.00000	.00000

TABLE 5.08

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .800   A=MEAN OF EXP DIST   M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
5.0251	.00033	.00065	.00098	.00131	.00164
4.7739	.00042	.00084	.00126	.00168	.00210
4.5226	.00054	.00108	.00161	.00215	.00269
4.2714	.00069	.00138	.00207	.00275	.00344
4.0201	.00088	.00176	.00264	.00352	.00441
4.0161	.00088	.00177	.00265	.00353	.00441
3.8153	.00107	.00215	.00322	.00430	.00537
3.7688	.00113	.00225	.00338	.00450	.00563
3.6145	.00131	.00261	.00392	.00522	.00653
3.5176	.00144	.00287	.00431	.00575	.00718
3.4137	.00159	.00317	.00476	.00634	.00793
3.2663	.00183	.00366	.00549	.00732	.00915
3.2129	.00192	.00385	.00577	.00769	.00962
3.0151	.00232	.00465	.00697	.00930	.01162
3.0120	.00233	.00466	.00698	.00931	.01164
3.0090	.00233	.00466	.00699	.00932	.01166
2.8586	.00269	.00537	.00806	.01074	.01343
2.8112	.00281	.00562	.00843	.01125	.01406
2.7638	.00294	.00588	.00883	.01177	.01471
2.7081	.00309	.00618	.00927	.01236	.01545
2.6104	.00339	.00677	.01016	.01355	.01693
2.5577	.00355	.00710	.01065	.01420	.01775
2.5126	.00371	.00741	.01112	.01483	.01854
2.4096	.00406	.00813	.01219	.01626	.02032
2.4072	.00407	.00814	.01221	.01628	.02034
2.2613	.00464	.00929	.01393	.01857	.02321
2.2568	.00465	.00930	.01396	.01861	.02326
2.2088	.00486	.00971	.01457	.01943	.02428
2.1063	.00530	.01061	.01591	.02121	.02652
2.0101	.00577	.01153	.01730	.02307	.02883
2.0080	.00577	.01154	.01731	.02309	.02886
2.0040	.00578	.01156	.01734	.02312	.02891
1.9559	.00602	.01205	.01807	.02410	.03012
1.9038	.00628	.01257	.01885	.02514	.03142
1.8072	.00681	.01362	.02042	.02723	.03404
1.8054	.00681	.01363	.02044	.02725	.03407
1.8036	.00682	.01364	.02045	.02727	.03409
1.7588	.00708	.01416	.02124	.02832	.03540
1.7034	.00738	.01476	.02214	.02952	.03690
1.6550	.00766	.01533	.02299	.03066	.03832
1.6064	.00795	.01591	.02386	.03182	.03977
1.6032	.00796	.01593	.02389	.03186	.03982
1.5075	.00856	.01711	.02567	.03422	.04278
1.5045	.00856	.01713	.02569	.03426	.04282
1.5030	.00857	.01714	.02571	.03428	.04284
1.4056	.00918	.01835	.02753	.03671	.04588
1.4028	.00919	.01837	.02756	.03674	.04593
1.3541	.00949	.01898	.02847	.03796	.04745
1.3026	.00980	.01961	.02941	.03921	.04902
1.2563	.01010	.02019	.03029	.04039	.05049

TABLE 5.08

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .800 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
1.2048	.01040	.02080	.03120	.04161	.05201
1.2036	.01041	.02081	.03122	.04162	.05203
1.2024	.01041	.02082	.03123	.04163	.05204
1.1022	.01098	.02197	.03295	.04393	.05492
1.0532	.01125	.02250	.03375	.04500	.05625
1.0050	.01150	.02299	.03449	.04598	.05748
1.0040	.01150	.02300	.03449	.04599	.05749
1.0020	.01150	.02301	.03451	.04601	.05751
1.0010	.01151	.02301	.03452	.04602	.05753
.9510	.01173	.02347	.03520	.04694	.05867
.9027	.01193	.02387	.03580	.04774	.05967
.9018	.01194	.02387	.03581	.04775	.05968
.9009	.01194	.02388	.03581	.04775	.05969
.8509	.01211	.02422	.03633	.04844	.06055
.8032	.01224	.02448	.03672	.04896	.06120
.8016	.01224	.02449	.03673	.04897	.06121
.8008	.01224	.02449	.03673	.04897	.06122
.7538	.01233	.02466	.03699	.04933	.06166
.7523	.01233	.02466	.03700	.04933	.06166
.7508	.01233	.02467	.03700	.04933	.06167
.7014	.01237	.02474	.03711	.04948	.06185
.7007	.01237	.02474	.03711	.04948	.06185
.6507	.01235	.02470	.03704	.04939	.06174
.6024	.01226	.02451	.03677	.04903	.06128
.6018	.01226	.02451	.03677	.04902	.06128
.6012	.01226	.02451	.03677	.04902	.06128
.6006	.01225	.02451	.03676	.04902	.06127
.5506	.01208	.02416	.03624	.04832	.06040
.5025	.01182	.02364	.03545	.04727	.05909
.5010	.01181	.02363	.03544	.04725	.05907
.5005	.01181	.02362	.03543	.04725	.05906
.4514	.01144	.02288	.03432	.04576	.05720
.4505	.01144	.02287	.03431	.04574	.05718
.4016	.01095	.02189	.03284	.04378	.05473
.4008	.01094	.02188	.03282	.04376	.05470
.4004	.01094	.02188	.03282	.04375	.05469
.3504	.01030	.02060	.03090	.04120	.05150
.3009	.00951	.01902	.02853	.03804	.04755
.3006	.00951	.01901	.02852	.03803	.04754
.3003	.00950	.01901	.02851	.03802	.04752
.2513	.00854	.01708	.02562	.03415	.04269
.2503	.00853	.01706	.02558	.03411	.04264
.2008	.00735	.01471	.02206	.02942	.03677
.2004	.00735	.01470	.02205	.02940	.03675
.2002	.00735	.01469	.02204	.02939	.03673
.1505	.00594	.01188	.01782	.02376	.02970
.1502	.00594	.01187	.01781	.02374	.02968
.1002	.00426	.00853	.01279	.01706	.02132
.1001	.00426	.00853	.01279	.01705	.02131
.0000	.00000	.00000	.00000	.00000	.00000

TABLE 5.09

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .900 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
5.0251	.00032	.00064	.00096	.00128	.00159
4.7739	.00041	.00082	.00123	.00164	.00205
4.5226	.00052	.00105	.00157	.00210	.00262
4.2714	.00067	.00134	.00201	.00269	.00336
4.0201	.00086	.00172	.00258	.00344	.00430
4.0161	.00086	.00172	.00258	.00344	.00430
3.8153	.00105	.00210	.00314	.00419	.00524
3.7688	.00110	.00220	.00329	.00439	.00549
3.6145	.00127	.00255	.00382	.00509	.00637
3.5176	.00140	.00280	.00420	.00560	.00701
3.4137	.00155	.00309	.00464	.00619	.00773
3.2663	.00178	.00357	.00535	.00714	.00892
3.2129	.00188	.00375	.00563	.00750	.00938
3.0151	.00227	.00453	.00680	.00907	.01133
3.0120	.00227	.00454	.00681	.00908	.01135
3.0090	.00227	.00455	.00682	.00909	.01137
2.8586	.00262	.00524	.00786	.01048	.01310
2.8112	.00274	.00548	.00822	.01097	.01371
2.7638	.00287	.00574	.00861	.01148	.01435
2.7081	.00301	.00603	.00904	.01205	.01507
2.6104	.00330	.00660	.00991	.01321	.01651
2.5577	.00346	.00692	.01038	.01385	.01731
2.5126	.00362	.00723	.01085	.01446	.01808
2.4096	.00396	.00793	.01189	.01585	.01982
2.4072	.00397	.00793	.01190	.01587	.01984
2.2613	.00453	.00905	.01358	.01811	.02264
2.2568	.00454	.00907	.01361	.01815	.02268
2.2088	.00474	.00947	.01421	.01894	.02368
2.1063	.00517	.01034	.01551	.02069	.02586
2.0101	.00562	.01125	.01687	.02249	.02811
2.0080	.00563	.01126	.01688	.02251	.02814
2.0040	.00564	.01127	.01691	.02255	.02819
1.9559	.00587	.01175	.01762	.02350	.02937
1.9038	.00613	.01226	.01838	.02451	.03064
1.8072	.00664	.01328	.01992	.02655	.03319
1.8054	.00664	.01329	.01993	.02657	.03322
1.8036	.00665	.01330	.01994	.02659	.03324
1.7588	.00690	.01381	.02071	.02762	.03452
1.7034	.00720	.01439	.02159	.02878	.03598
1.6550	.00747	.01495	.02242	.02989	.03737
1.6064	.00776	.01551	.02327	.03103	.03878
1.6032	.00777	.01553	.02330	.03106	.03883
1.5075	.00834	.01668	.02503	.03337	.04171
1.5045	.00835	.01670	.02505	.03340	.04176
1.5030	.00836	.01671	.02507	.03342	.04178
1.4056	.00895	.01790	.02684	.03579	.04474
1.4028	.00896	.01791	.02687	.03583	.04478
1.3541	.00925	.01851	.02776	.03702	.04627
1.3026	.00956	.01912	.02868	.03824	.04779
1.2563	.00985	.01969	.02954	.03938	.04923

TABLE 5.C9

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A= .900   A=MEAN OF EXP DIST   M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
1.2048	.01014	.02028	.03043	.04057	.05071
1.2036	.01015	.02029	.03044	.04058	.05073
1.2024	.01015	.02030	.03045	.04060	.05075
1.1022	.01071	.02142	.03213	.04284	.05355
1.0532	.01097	.02194	.03291	.04388	.05484
1.0050	.01121	.02242	.03363	.04484	.05605
1.0040	.01121	.02242	.03364	.04485	.05606
1.0020	.01122	.02243	.03365	.04487	.05608
1.0010	.01122	.02244	.03366	.04488	.05609
.9510	.01144	.02289	.03433	.04577	.05721
.9027	.01164	.02327	.03491	.04655	.05819
.9018	.01164	.02328	.03492	.04656	.05820
.9009	.01164	.02328	.03492	.04656	.05820
.8509	.01181	.02362	.03542	.04723	.05904
.8032	.01194	.02387	.03581	.04774	.05968
.8016	.01194	.02388	.03581	.04775	.05969
.8008	.01194	.02388	.03582	.04776	.05969
.7538	.01202	.02405	.03607	.04810	.06012
.7523	.01203	.02405	.03608	.04810	.06013
.7508	.01203	.02405	.03608	.04810	.06013
.7014	.01206	.02413	.03619	.04825	.06031
.7007	.01206	.02413	.03619	.04825	.06031
.6507	.01204	.02408	.03612	.04816	.06020
.6024	.01195	.02390	.03586	.04781	.05976
.6018	.01195	.02390	.03585	.04780	.05975
.6012	.01195	.02390	.03585	.04780	.05975
.6006	.01195	.02390	.03585	.04780	.05975
.5506	.01178	.02356	.03534	.04712	.05889
.5025	.01152	.02305	.03457	.04609	.05762
.5010	.01152	.02304	.03456	.04608	.05760
.5005	.01152	.02303	.03455	.04607	.05759
.4514	.01116	.02231	.03347	.04462	.05578
.4505	.01115	.02230	.03345	.04461	.05576
.4016	.01067	.02135	.03202	.04269	.05336
.4008	.01067	.02134	.03200	.04267	.05334
.4004	.01067	.02133	.03200	.04266	.05333
.3504	.01004	.02009	.03013	.04018	.05022
.3009	.00927	.01855	.02782	.03709	.04636
.3006	.00927	.01854	.02781	.03708	.04635
.3003	.00927	.01854	.02780	.03707	.04634
.2513	.00833	.01665	.02498	.03330	.04163
.2503	.00832	.01663	.02495	.03326	.04158
.2008	.00717	.01434	.02151	.02869	.03586
.2004	.00717	.01433	.02150	.02867	.03583
.2002	.00716	.01433	.02149	.02866	.03582
.1505	.00579	.01158	.01738	.02317	.02896
.1502	.00579	.01157	.01736	.02315	.02894
.1002	.00416	.00832	.01248	.01663	.02079
.1001	.00416	.00831	.01247	.01663	.02078
.0000	.00000	.00000	.00000	.00000	.00000

TABLE 5.10

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A=1.000 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
5.0251	.00031	.00061	.00092	.00123	.00154
4.7739	.00039	.00079	.00118	.00158	.00197
4.5226	.00051	.00101	.00152	.00202	.00253
4.2714	.00065	.00129	.00194	.00259	.00324
4.0201	.00083	.00166	.00248	.00331	.00414
4.0161	.00083	.00166	.00249	.00332	.00415
3.8153	.00101	.00202	.00303	.00404	.00505
3.7688	.00106	.00212	.00317	.00423	.00529
3.6145	.00123	.00245	.00368	.00491	.00614
3.5176	.00135	.00270	.00405	.00540	.00675
3.4137	.00149	.00298	.00447	.00596	.00745
3.2663	.00172	.00344	.00516	.00688	.00860
3.2129	.00181	.00361	.00542	.00723	.00904
3.0151	.00218	.00437	.00655	.00874	.01092
3.0120	.00219	.00438	.00656	.00875	.01094
3.0090	.00219	.00438	.00657	.00876	.01095
2.8586	.00252	.00505	.00757	.01010	.01262
2.8112	.00264	.00528	.00793	.01057	.01321
2.7638	.00277	.00553	.00830	.01106	.01383
2.7081	.00290	.00581	.00871	.01162	.01452
2.6104	.00318	.00637	.00955	.01273	.01591
2.5577	.00334	.00667	.01001	.01334	.01668
2.5126	.00348	.00697	.01045	.01394	.01742
2.4096	.00382	.00764	.01146	.01528	.01910
2.4072	.00382	.00765	.01147	.01530	.01912
2.2613	.00436	.00873	.01309	.01745	.02182
2.2568	.00437	.00874	.01312	.01749	.02186
2.2088	.00456	.00913	.01369	.01826	.02282
2.1063	.00498	.00997	.01495	.01994	.02492
2.0101	.00542	.01084	.01626	.02168	.02710
2.0080	.00542	.01085	.01627	.02170	.02712
2.0040	.00543	.01087	.01630	.02173	.02717
1.9559	.00566	.01132	.01699	.02265	.02831
1.9038	.00591	.01181	.01772	.02363	.02953
1.8072	.00640	.01280	.01920	.02559	.03199
1.8054	.00640	.01281	.01921	.02561	.03202
1.8036	.00641	.01282	.01922	.02563	.03204
1.7588	.00665	.01331	.01996	.02662	.03327
1.7034	.00694	.01387	.02081	.02774	.03468
1.6550	.00720	.01441	.02161	.02881	.03601
1.6064	.00748	.01495	.02243	.02990	.03738
1.6032	.00749	.01497	.02246	.02994	.03743
1.5075	.00804	.01608	.02412	.03216	.04020
1.5045	.00805	.01610	.02415	.03220	.04025
1.5030	.00805	.01611	.02416	.03221	.04027
1.4056	.00862	.01725	.02587	.03450	.04312
1.4028	.00863	.01727	.02590	.03453	.04316
1.3541	.00892	.01784	.02676	.03568	.04460
1.3026	.00921	.01843	.02764	.03685	.04607
1.2563	.00949	.01898	.02847	.03796	.04745

TABLE 5.10

TABLE OF JOINT EXPONENTIAL,GEOMETRIC RELIABILITY DIFFERENCES

T/A=1.000 A=MEAN OF EXP DIST M=MEAN OF GEOM DIST

K/M	RHO=.05	.10	.15	.20	.25
1.2048	.00978	.01955	.02933	.03910	.04888
1.2036	.00978	.01956	.02934	.03912	.04889
1.2024	.00978	.01956	.02935	.03913	.04891
1.1022	.01032	.02064	.03097	.04129	.05161
1.0532	.01057	.02114	.03172	.04229	.05286
1.0050	.01080	.02161	.03241	.04322	.05402
1.0040	.01081	.02161	.03242	.04323	.05403
1.0020	.01081	.02162	.03243	.04324	.05405
1.0010	.01081	.02163	.03244	.04325	.05407
.9510	.01103	.02206	.03309	.04411	.05514
.9027	.01122	.02243	.03365	.04487	.05608
.9018	.01122	.02244	.03365	.04487	.05609
.9009	.01122	.02244	.03366	.04488	.05610
.8509	.01138	.02276	.03414	.04552	.05690
.8032	.01150	.02301	.03451	.04602	.05752
.8016	.01151	.02301	.03452	.04602	.05753
.8008	.01151	.02301	.03452	.04603	.05753
.7538	.01159	.02318	.03477	.04636	.05795
.7523	.01159	.02318	.03477	.04636	.05795
.7508	.01159	.02318	.03477	.04636	.05796
.7014	.01163	.02325	.03488	.04651	.05813
.7007	.01163	.02325	.03488	.04651	.05813
.6507	.01160	.02321	.03481	.04642	.05802
.6024	.01152	.02304	.03456	.04608	.05760
.6018	.01152	.02304	.03456	.04607	.05759
.6012	.01152	.02304	.03455	.04607	.05759
.6006	.01152	.02303	.03455	.04607	.05759
.5506	.01135	.02271	.03406	.04541	.05676
.5025	.01111	.02221	.03332	.04443	.05553
.5010	.01110	.02220	.03331	.04441	.05551
.5005	.01110	.02220	.03330	.04440	.05550
.4514	.01075	.02150	.03226	.04301	.05376
.4505	.01075	.02150	.03224	.04299	.05374
.4016	.01029	.02057	.03086	.04115	.05143
.4008	.01028	.02056	.03085	.04113	.05141
.4004	.01028	.02056	.03084	.04112	.05140
.3504	.00968	.01936	.02904	.03872	.04840
.3009	.00894	.01787	.02681	.03575	.04469
.3006	.00893	.01787	.02680	.03574	.04467
.3003	.00893	.01786	.02680	.03573	.04466
.2513	.00802	.01605	.02407	.03210	.04012
.2503	.00801	.01603	.02404	.03206	.04007
.2008	.00691	.01382	.02074	.02765	.03456
.2004	.00691	.01381	.02072	.02763	.03454
.2002	.00690	.01381	.02071	.02762	.03452
.1505	.00558	.01116	.01675	.02233	.02791
.1502	.00558	.01116	.01673	.02231	.02789
.1002	.00401	.00802	.01202	.01603	.02004
.1001	.00401	.00801	.01202	.01603	.02003
.0000	.00000	.00000	.00000	.00000	.00000

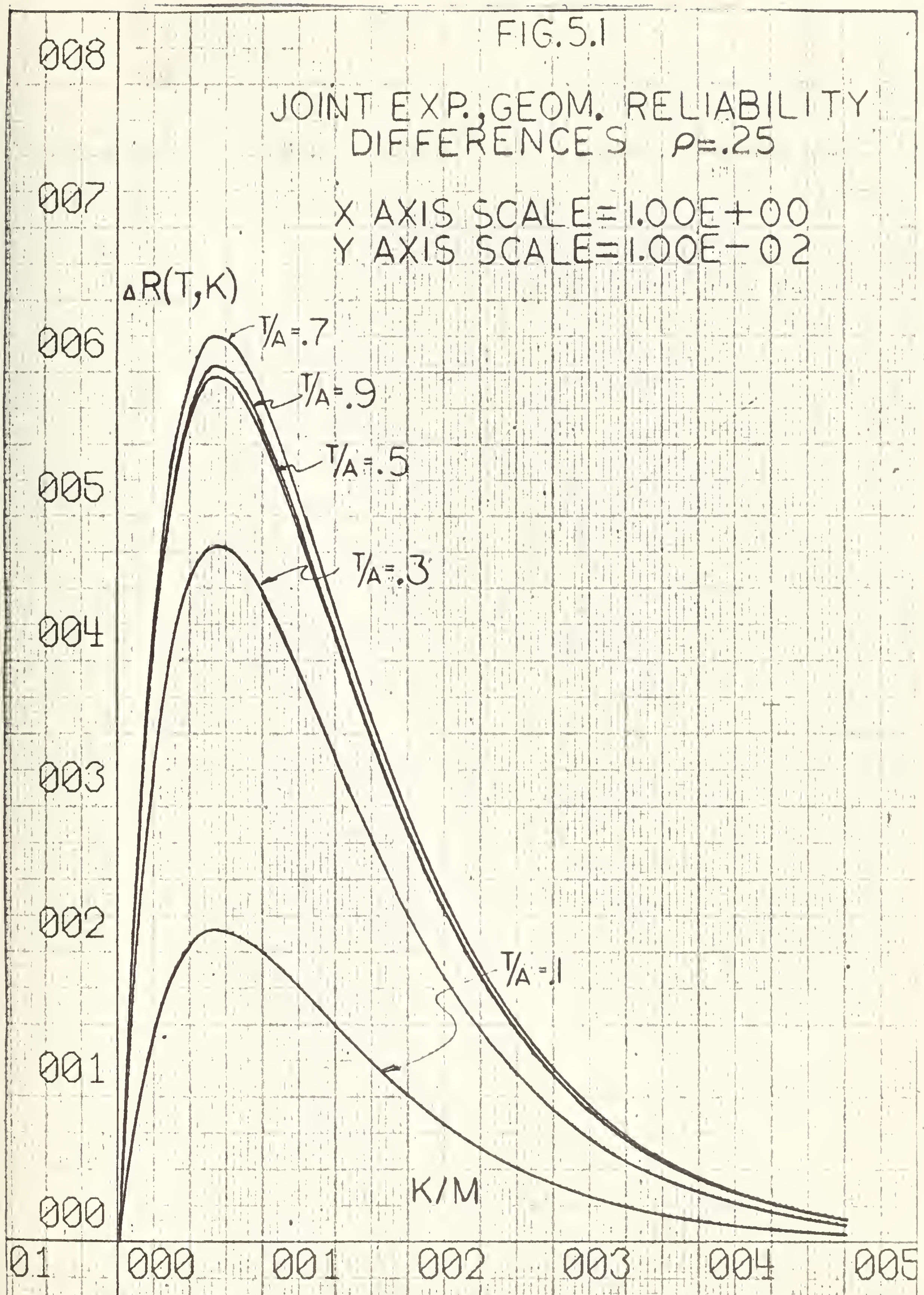
FIG.5.1
JOINT EXP. GEOM. RELIABILITY
DIFFERENCES ρ=.25
X AXIS SCALE=1.00E+00
Y AXIS SCALE=1.00E-02
ΔR(T,K)
T/A=.7
T/A=.9
T/A=.5
T/A=.3
T/A=.1
K/M
008
007
006
005
004
003
002
001
000
01
000
001
002
003
004
005

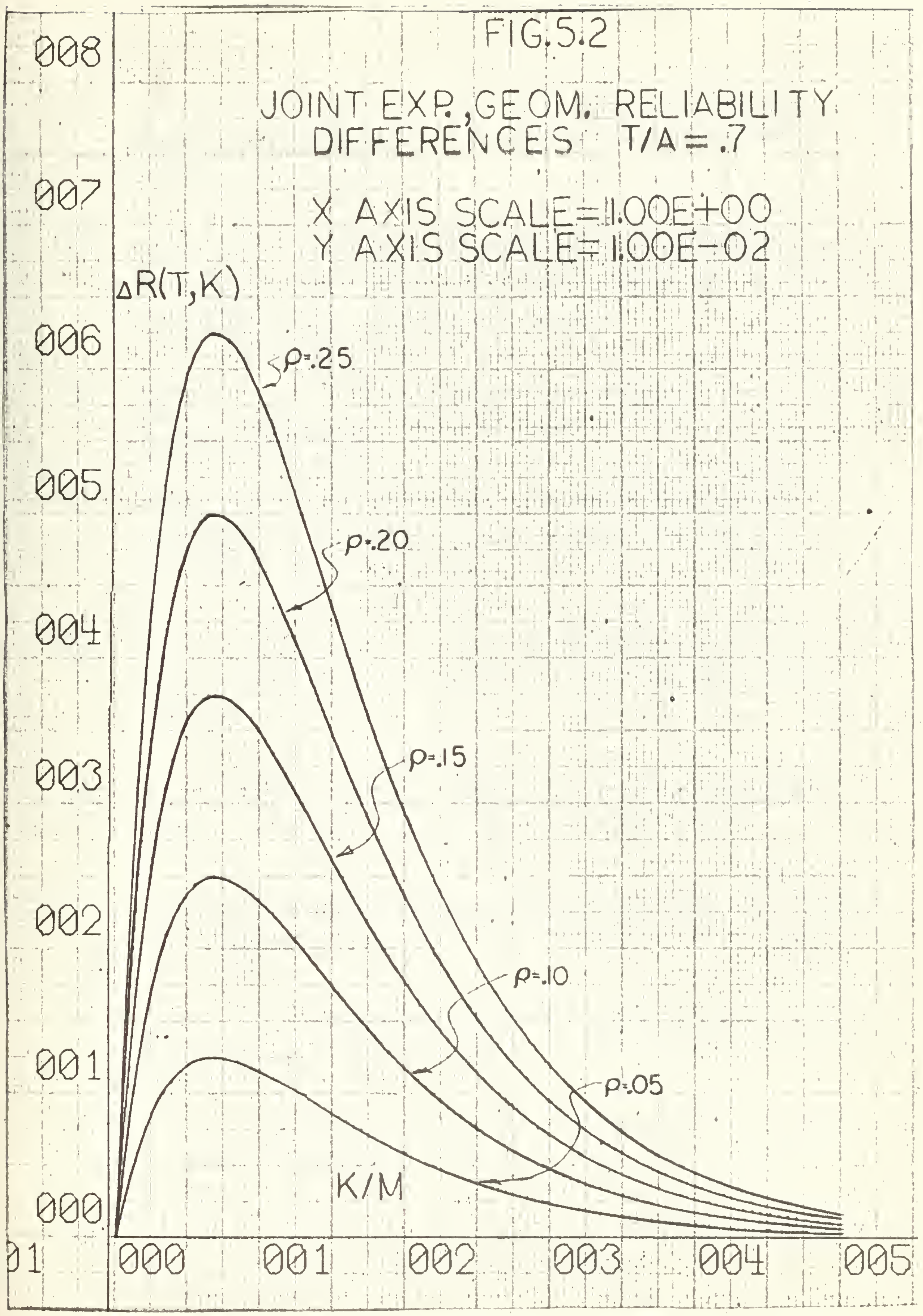
FIG.5.2
JOINT EXP.,GEOM. RELIABILITY
DIFFERENCES T/A = .7
X AXIS SCALE=1.00E+00
Y AXIS SCALE=1.00E-02
ΔR(T,K)
ρ=.25
ρ=.20
ρ=.15
ρ=.10
ρ=.05
K/M
008
007
006
005
004
003
002
001
000
000
001
002
003
004
005

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APPENDIX A.1

MATHEMATICAL DEVELOPMENT

BIVARIATE EXPONENTIAL DISTRIBUTION

To show $f_{XY}(x,y)$ is a density function:

must show: (1) $f_{XY}(x,y) \geq 0$

$$(2) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$(3) \quad \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = f_Y(y) \text{ and}$$

$$\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = f_X(x)$$

where

$$f_{XY}(x,y) = \frac{1}{ab} e^{-(x/a+y/b)}\left[1 + v(1-2e^{-x/a})(1-2e^{-y/b})\right]$$

(1) with $a>0$, $b>0$, $-1\leq v\leq 1$, $x\geq 0$, and $y\geq 0$

it can be seen that $-1\leq 1 - 2e^{-x/a}\leq 1$

since $0\leq 2e^{-x/a}\leq 2$

therefore since $-1\leq v\leq 1$

$$-1\leq v(1 - 2e^{-x/a})(1 - 2e^{-y/b})\leq 1$$

and $$1 + v(1 - 2e^{-x/a})(1 - 2e^{-y/b})\geq 0$$

now $\frac{1}{ab} e^{-(x/a + y/b)} \geq 0$

since $0 \leq e^{-(x/a + y/b)} \leq 1$ and $\frac{1}{ab} > 0$

Therefore the product

$$f_{XY}(x,y) = \frac{1}{ab} e^{-(x/a + y/b)} \left[1 + v(1-2e^{-x/a})(1-2e^{-y/b})\right] \geq 0$$

$$(2) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dx\,dy = \int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{ab} e^{-(x/a + y/b)} \left[1 + v(1 - 2e^{-x/a})(1 - 2e^{-y/b})\right] dx\,dy$$

$$= \frac{1}{ab} \int_{0}^{\infty} e^{-x/a}dx \int_{0}^{\infty} e^{-y/b}dy + \frac{v}{ab} \left[\int_{0}^{\infty} e^{-x/a}(1-2e^{-x/a})dx \int_{0}^{\infty} e^{-y/b}(1-2e^{-y/b})dy\right]$$

look at $\frac{1}{a} \int_{0}^{\infty} e^{-x/a}dx$, $\frac{1}{b} \int_{0}^{\infty} e^{-y/b}dy$

$$\frac{1}{a} \int_{0}^{\infty} e^{-x/a}dx = \frac{-a}{a} \left[e^{-x/a} \right]_{0}^{\infty} = 1$$

similarly $\frac{1}{b} \int_{0}^{\infty} e^{-y/b}dy = 1$

look at $\frac{1}{a} \int_{0}^{\infty} e^{-x/a}(1 - 2e^{-x/a})dx$

let $u = (1 - 2e^{-x/a})$ then $du = \frac{2}{a} e^{-x/a}$

and for $x = 0$, $u = 1$; for $x \to \infty$, $u = -1$

hence $\frac{1}{a} \int_{0}^{\infty} e^{-x/a}(1-2e^{-x/a})dx = \frac{1}{a} \int_{-1}^{1} 2u\,du = \left.\frac{u^2}{a}\right]_{-1}^{1} = 0$

Then $\quad E(xy) = \int_0^\infty \int_0^\infty xy\, f_{XY}(x,y)dx\, dy$

$$E(xy) = \int_0^\infty \frac{x}{a} e^{-x/a}dx \int_0^\infty \frac{y}{b} e^{-y/b}dy + v \left[\int_0^\infty \frac{x}{a} e^{-x/a}(1-2e^{-x/a})dx \right.$$
$$\left. \int_0^\infty \frac{y}{b} e^{-y/b}(1-2e^{-y/b})dy \right]$$

look at $\int_0^\infty \frac{x}{a} e^{-x/a}dx$; integrating by parts

where $\quad u = x/a \quad du = \frac{1}{a} dx \quad dv = e^{-x/a}dx \quad v = -a\, e^{-x/a}$

then $\int_0^\infty \frac{x}{a} e^{-x/a}dx = (\frac{x}{a})(-a\, e^{-x/a})\Big]_0^\infty - \int_0^\infty -\frac{a}{a} e^{-x/a}dx$

$\int_0^\infty x/a\, e^{-x/a}dx = 0 + a = a \quad$ similarly $\int_0^\infty y/b\, e^{-y/b}dy = b$

Also, if we expand the second term we get

$$\int_0^\infty x/a\, e^{-x/a}dx - \int_0^\infty 2x/a\, e^{-2x/a}dx$$

and if we substitute 2/a for 1/a in the integration by parts, we

obtain $\quad \int_0^\infty 2x/a\, e^{-2x/a}dx = a/2$

Hence from

$$E(xy) = \int_0^\infty x/a\, e^{-x/a}dx \int_0^\infty y/b\, e^{-y/b}dy + v \left[\left[\int_0^\infty x/a\, e^{-x/a}dx - \int_0^\infty 2x/a\, e^{-x/a}dx \right] \right.$$
$$\left. \left[\int_0^\infty y/b\, e^{-y/b}dy - \int_0^\infty 2y/b\, e^{-2y/b}dy \right] \right]$$

similarly

$$\frac{1}{b}\int_0^\infty e^{-y/b}(1-2e^{-y/b})dy = \frac{1}{b}\int_{-1}^{1} 2\,u\,du = \frac{u^2}{b}\Bigg]_{-1}^{1} = 0$$

Therefore

$$\int_0^\infty\int_0^\infty f_{XY}(x,y)\,dx\,dy = 1\cdot1 + v\left[0\cdot0\right] = 1$$

(3)

$$\int_0^\infty f_{XY}(x,y)dx = \int_0^\infty \frac{1}{ab}\,e^{-(x/a+y/b)}dx + \int_0^\infty \frac{v}{ab}\,e^{-(x/a+y/b)}(1-2e^{-x/a})(1-2e^{-y/b})dx$$

$$= \frac{1}{ab}\,e^{-y/b}\int_0^\infty e^{-x/a}dx + \frac{v}{ab}\,e^{-y/b}(1-2e^{-y/b})\int_0^\infty e^{-x/a}(1-2e^{-x/a})dx$$

$$= \frac{1}{ab}\,e^{-y/b}(a) + \frac{v}{ab}\,e^{-y/b}(1-2e^{-y/b})\quad(0)$$

hence $\displaystyle\int_0^\infty f_{XY}(x,y)dx = \frac{1}{b}\,e^{-y/b} = f_Y(y)$

similarly $\displaystyle\int_0^\infty f_{XY}(x,y)\,dy = \frac{1}{a}\,e^{-x/a} = f_X(x)$

To evaluate the correlation coefficient:

We know: $E(x) = a,\ E(y) = b,\ \sigma_x = a,\ \sigma_y = b$

and $\displaystyle\rho = \frac{E(xy) - E(x)\,E(y)}{\sigma_x\sigma_y}$

we obtain

$$E(xy) = ab + v\left(\frac{a}{2} \cdot \frac{b}{2}\right) = ab\,(1 + v/4)$$

Therefore

$$\rho = \frac{ab\,(1 + v/4) - a \cdot b}{a\,b}$$

$$\rho = v/4$$

Since $|v| \leq 1$ then $|\rho| \leq 1/4$

To derive the reliability function:

$$R(t) = P\left[x \geq t,\ y \geq t\right] = \int_t^{\infty}\int_t^{\infty} f_{XY}(x,y)\,dx\,dy$$

$$R(t) = \int_t^{\infty} \frac{1}{a} e^{-x/a}dx \int_t^{\infty} \frac{1}{b} e^{-y/b}dy + v\left[\int_t^{\infty} \frac{1}{a} e^{-x/a}(1-2e^{-x/a})dx \right.$$

$$\left.\int_t^{\infty} \frac{1}{b} e^{-y/b}(1-2e^{-y/b})dy\right]$$

look at $\displaystyle\int_t^{\infty} \frac{1}{a}\, e^{-x/a}dx = \frac{-a}{a}\left[e^{-x/a}\right]_t^{\infty} = e^{-t/a}$

using similar techniques as for the previous integrations we obtain

$$R(t) = (e^{-t/a})(e^{-t/b}) + v\left[(e^{-t/a} - e^{-2t/a})(e^{-t/b} - e^{-2t/b})\right]$$

$$\therefore R(t) = e^{-(t/a+t/b)}\left[1 + v\,(1-e^{-t/a})(1-e^{-t/b})\right]$$

To derive the reliability difference function:

$$R_2\,(t) = e^{-(t/a+t/b)}\left[1 + v(1-e^{-t/a})(1-e^{-t/b})\right]$$

$$R_1\,(t) = e^{\,(t/a + t/b)}$$

hence since $\Delta R(t) = R_2\,(t) - R_1\,(t)$ then

$$\Delta R(t) = v\,e^{-(t/a + t/b)}\,(1 - e^{-t/a})(1 - e^{-t/b})$$

$$\therefore R(t) = 4\rho\,e^{-(t/a + t/b)}\,(1 - e^{-t/a})(1 - e^{-t/b})$$

To find the point where $\Delta R(t)$ is a maximum with respect to t/a:

let a = b so that

$$\Delta R(t) = 4\rho e^{-2t/a}\,(1 - e^{-t/a})^2$$

$$\frac{\delta \Delta R(t)}{\delta\,(t/a)} = 4\rho\left[-2\,e^{-2t/a}(1-e^{-t/a})^2 + 2e^{-2t/a}(1-e^{-t/a})e^{-t/a}\right] = 0$$

hence $-\,(1-e^{-t/a}) + e^{-t/a} = 0,\qquad e^{-t/a} = \frac{1}{2}$

$$t/a\,\Big]_{max} = -\ln\frac{1}{2} = .69315$$

APPENDIX A.2

MATHEMATICAL DEVELOPMENT

BIVARIATE GEOMETRIC DISTRIBUTION

To show that $\sum_{x=0}^{k} p^x q = 1 - p^{k+1}$

$$\sum_{x=0}^{k} p^x q = q \sum_{x=0}^{k} p^x = q \left[\sum_{x=0}^{\infty} p^x - \sum_{x=k+1}^{\infty} p^x \right]$$

note: $\sum_{x=0}^{\infty} p^x = \frac{1}{1-p}$ geometric series and converges since $|p| < 1$

therefore

$$q \left[\sum_{x=0}^{\infty} p^x - \sum_{x=k+1}^{\infty} p^x \right]$$

$$= q \left[\frac{1}{1-p} - \left(p^{k+1} + p^{k+2} + p^{k+3} + \cdots \right) \right]$$

$$= q \left[\frac{1}{1-p} - p^{k+1} \left(1 + p + p^2 + p^3 \cdots \right) \right]$$

$$= q \left[\frac{1}{1-p} - p^{k+1} \sum_{i=0}^{\infty} p^i \right]$$

$$= q \left[\frac{1}{1-p} - p^{k+1} \frac{1}{1-p} \right]$$

$$= q \left[\frac{1 - p^{k+1}}{1 - p} \right] = 1 - p^{k+1}$$

Lemma 1

Since the geometric series $\sum_{i=0}^{\infty} x^i$ converges to $\frac{1}{1-x}$, provided

$|x|<1$, the product $\frac{1}{1-x} \cdot \frac{1}{1-x}$ can be shown, by multiplying the

series expansions, to be

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

if we multiply both sides by x we obtain

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

but this is $\sum_{i=0}^{\infty} ix^i$

and therefore the series $\sum_{i=0}^{\infty} ix^i$ converges to $\frac{x}{(1-x)^2}$, provided

$|x| < 1$

To show that $f_{XY}(x,y)$ is a probability mass function

when

$$f_{XY}(x,y)=f_X(x)\, f_Y(y)\left[1+v \left[2F_X(x)-f_X(x)-1\right]\left[2F_Y(y)-f_Y(y)-1\right]\right]$$

where $f_X(x) = p^x q$ $\qquad F_X(x) = 1 - p^{x+1}$

$f_Y(y) = p^y q$ $\qquad F_Y(y) = 1 - p^{y+1}$

with $0 < p < 1$ $\qquad q = 1 - p$ $\qquad 0 \leq v \leq 1$

$x = 0, 1, 2, \cdots$ $\qquad y = 0, 1, 2, \cdots$

we must show

(1) $f_{XY}(x,y) \geq 0$

(2) $\sum_{x=0}^{\infty} \sum_{y=0}^{\infty} f_{XY}(x,y) = 1$

(3) $\sum_{x=0}^{\infty} f_{XY}(x,y) = f_Y(y) \quad , \quad \sum_{y=0}^{\infty} f_{XY}(x,y) = f_X(x)$

---

(1) $f_{XY}(x,y) = p_1^x q_1 p_2^y q_2 \left[1 + v\left[2(1-p_1^{x+1}) - p_1^x q_1 - 1\right]\left[2(1-p_2^{y+1}) - p_2^y q_2 - 1\right]\right]$

$= p_1^x q_1 p_2^y q_2 \left[1 + v\left[1 - 2p_1^{x+1} - p_1^x q_1 \quad 1 - 2p_2^{y+1} - p_2^y q_2\right]\right]$

$= p_1^x q_1 p_2^y q_2 \left[1 + v\left[1 - p_1^{x+1} - p_1^x\right]\left[1 - p_2^{y+1} - p_2^y\right]\right]$

looking at $1 - p^{x+1} - p^x$

since

$$1 \leq p + 1 \leq 2$$

$$0 \leq p^x(p + 1) \leq 2$$

$$0 - 1 \leq p^x(p + 1) - 1 \leq 2 - 1$$

$$1 \geq 1 - p^x(p + 1) \geq - 1$$

then calling

$$\left[1 - p_1^{x+1} - p_1^x\right]\left[1 - p_2^{y+1} - p_2^y\right] = N_X N_Y$$

it can be seen that

$$1 \geq N_X N_Y \geq - 1$$

Therefore

$1 + v N_X N_Y \geq 0$ since $-1 \leq v \leq 1$ also

now since $p_1^x q_1 p_2^y q_2 \geq 0$ for all x, y, p_1, p_2 then the

product

$$p_1^x q_1 p_2^y q_2 (1 + v N_X N_Y) \geq 0$$

hence

$$f_{XY}(x,y) = p_1^{x} q_1 p_2^{y} q_2 (1 + v N_X N_Y) \geq 0$$

---

(2)
To show that the bivariate geometric distribution sums to 1

$$f_{XY}(x,y) = f_X(x) f_Y(y)\left[1+v\left[2F_X(x)-f_X(x)-1\right]\left[2F_Y(y)-f_Y(y)-1\right]\right]$$

where $f_X(x) = p^x q$ $\qquad F_X(x) = 1 - p^{x+1}$

$fY(y) = p^y q$ $\qquad F_Y(y) = 1 - p^{y+1}$

$$\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} f_{XY}(x,y)=\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\left[p_1^{x}q_1p_2^{y}q_2\left[1+v\left[2(1-p_1^{x+1})-p_1^{x}q_1-1\right]\right.\right.$$
$$\left.\left.\left[2(1-p_2^{y+1})-p_2^{y}q_2-1\right]\right]\right]$$

$$=\sum_{z=0}^{\infty}\sum_{y=0}^{\infty}\left[p_1^{x}q_1p_2^{y}q_2+v\left[p_1^{x}q_1\left[1-2p_1^{x+1}-p_1^{x}q_1\right]p_2^{y}q_2\left[1-2p_2^{y+1}-p_2^{y}q_2\right]\right]\right]$$

$$=\sum_{x=0}^{\infty}p_1^{x}q_1\sum_{y=0}^{\infty}p_2^{y}q_2+v\left[\sum_{x=0}^{\infty}\left[p_1^{x}q_1-2p_1^{2x+1}q_1-p_1^{2x}q_1^{2}\right]\right.$$
$$\left.\sum_{y=0}^{\infty}\left[p_2^{y}q_2-2p_2^{2y+1}q_2-p_2^{2y}q_2^{2}\right]\right] \qquad \text{(A)}$$

looking at this term by term

$$\sum_{x=0}^{\infty}p^{x}q = q\sum_{x=0}^{\infty}p^{x} = q\frac{1}{1-p} = q/q = 1$$

looking at

$$\sum_{x=0}^{\infty} \left[p^x q - 2p^{2x+1} q - p^{2x} q^2 \right]$$

$$= \sum_{x=0}^{\infty} p^x q - 2 \sum_{x=0}^{\infty} p^{2x+1} q - \sum_{x=0}^{\infty} p^{2x} q^2$$

$$= 1 - 2pq \sum_{x=0}^{\infty} (p^2)^x - q^2 \sum_{x=0}^{\infty} (p^2)^x$$

$$= 1 - 2pq \frac{1}{1-p^2} - q^2 \frac{1}{1-p^2} \qquad \text{since } p^2 < 1$$

$$= 1 - \frac{2p}{1+p} - \frac{q}{1+p}$$

$$= \frac{1+p-2p-(1-p)}{1+p}$$

$$= \frac{1+p-2p-1+p}{1+p} = 0$$

Therefore putting these values into equation (A) gives

$$\sum_{x=0}^{\infty} \sum_{y=0}^{\infty} f_{XY}(x,y) = 1 \cdot 1 + v \;\; 0 \cdot 0 \;\; = 1$$

---

(3) To show that $\sum_{x=0}^{\infty} f_{XY}(x,y) = f_Y(y)$

from equation (A)

$$\sum_{x=0}^{\infty} f_{XY}(x,y) = \sum_{x=0}^{\infty} p_1{}^x q_1\ p_2{}^y q_2 + v\left[\sum_{x=0}^{\infty}\left[p_1{}^x q_1 - 2p_1{}^{2x+1} q_1 - p_1{}^{2x} q_1{}^2\right]\right.$$

$$\left.\left[p_2{}^y q_2 - 2p_2{}^{2y+1} q_2 - p_2{}^{2y} q_2{}^2\right]\right]$$

have already shown

(a) $\sum_{x=0}^{\infty} p^x q = 1$

(b) $\sum_{x=0}^{\infty} \left[p^x q - 2p^{2x+1} q - p^{2x} q^2\right] = 0$

simply substituting these into above equation

$$\sum_{x=0}^{\infty} f_{XY}(x,y) = p_2{}^y q_2 = f_Y(y)$$

It can be seen that by a similar argument

$$\sum_{y=0}^{\infty} f_{XY}(x,y) = f_X(x)$$

To compute E(x,y) and ρ

$$E(x,y) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} x\ y\ f_{XY}(x,y)$$

from equation (A) and the equation preceeding it

$$= \sum_{x=0}^{\infty} x\, p_1^x q_1 \sum_{y=0}^{\infty} y\, p_2^y q_2 + v \left[\sum_{x=0}^{\infty} x \left[p_1^x q_1 - 2p_1^{2x+1} q_1 - p_1^{2x} q_1^2 \right] \right.$$

$$\left. \sum_{y=0}^{\infty} y \left[p_2^y q_2 - 2p_2^{2y+1} q_2 - p_2^{2y} q_2^2 \right] \right] \qquad \text{(B)}$$

looking at this term by term

$$\sum_{x=0}^{\infty} x\, p^x q = q \sum_{x=0}^{\infty} x p^x$$

$$= q \frac{p}{(1-p)^2} \qquad \text{(using Lemma 1)}$$

$$= \frac{q\, p}{q^2} = \frac{p}{q}$$

looking at

$$\sum_{x=0}^{\infty} x \left[p^x q - 2p^{2x+1} q - p^{2x} q^2 \right]$$

$$= \sum_{x=0}^{\infty} x\, p^x q - 2 \sum_{x=0}^{\infty} x\, p^{2x+1} q - \sum_{x=0}^{\infty} x\, p^{2x} q^2$$

$$= \frac{p}{q} - 2pq \sum_{x=0}^{\infty} x (p^2)^x - q^2 \sum_{x=0}^{\infty} x (p^2)^x$$

making use of Lemma 1 and the fact $p^2 < 1$

we have

$$\sum_{x=0}^{\infty} x \left[p^x q - 2p^{2x+1} q - p^{2x} q^2\right]$$

$$= \frac{p}{q} - 2pq \frac{p^2}{(1-p^2)^2} - q^2 \frac{p^2}{(1-p^2)^2}$$

$$= \frac{p}{q} - \frac{2q\, p^3}{(1-p)^2(1+p)^2} - \frac{q^2\, p^2}{(1-p)^2(1+p)^2}$$

$$= \frac{p}{q} - \frac{2p^3}{q(1+p)^2} - \frac{p^2}{(1+p)^2}$$

$$= \frac{p(1+p)^2 - 2p^3 - qp^2}{q(1+p)^2} = \frac{p + 2p^2 + p^3 - 2p^3 - p^2 + p^3}{q(1+p)^2}$$

$$= \frac{p + p^2}{q(1+p)^2} = \frac{p(1+p)}{q(1+p)^2} = \frac{p}{q(1+p)}$$

Putting these values into equation (B) gives:

$$E(x,y) = \left(\frac{p_1}{q_1}\right)\left(\frac{p_2}{q_2}\right) + v\left[\frac{p}{q_1(1+p_1)} \quad \frac{p_2}{q_2(1+p_2)}\right]$$

$$E(x,y) = \frac{p_1\ p_2}{q_1\ q_2}\left[1 + \frac{v}{(1+p_1)(1+p_2)}\right]$$

To compute ρ

$$E(x) = \sum_{x=0}^{\infty} x\, p^x q = q \sum_{x=0}^{\infty} x\, p^x = \frac{qp}{(1-p)^2} = \frac{p}{q}$$

Making use of Lemma 1

Lloyd & Lipow [1] p 123 gives $\sigma_x^2 = \dfrac{p}{q^2}$ for this distribution,

therefore $\sigma_x = \dfrac{p}{q}$

now using the defining equation for ρ i.e.

$$\rho = \frac{E(x,y) - E(x)\,E(y)}{\sigma_x \sigma_y}$$

gives

$$\rho = \frac{\dfrac{p_1\,p_2}{q_1\,q_2}\left[1 + \dfrac{v}{(1+p_1)(1+p_2)}\right] - \dfrac{p_1}{q_1}\;\dfrac{p_2}{q_2}}{\dfrac{\sqrt{p_1}}{q_1}\;\dfrac{\sqrt{p_2}}{q_2}}$$

$$= \frac{p_2\,p_2 \quad 1 + \left[\dfrac{v}{(1+p_1)\,(1+p_2)}\right]}{\sqrt{p_1\,p_2}} = \sqrt{p_1\,p_2}\;\left[\frac{v}{(1+p_1)\,(1+p_2)}\right]$$

$$\rho = \frac{v\sqrt{p_1\,p_2}}{(1+p_1)\,(1+p_2)}$$

since $|v| \leq 1$

$0 < p_1$ and $p_2 < 1$

then $-\frac{1}{4} \leq \rho \leq \frac{1}{4}$

To compute $R(k_o)$

$$R(k_o) = P\left[X \geq k_o,\ Y \geq k_o\right] = \sum_{x=k_o}^{\infty} \sum_{y=k_o}^{\infty} f_{XY}(x,y)$$

$$= \sum_{x=k_o}^{\infty} \sum_{y=k_o}^{\infty} \left[f_X(x)\, f_Y(y) \left(1+v\left[2F_X(x)-f_X(x)-1\right] \left[2F_Y(y)-f_Y(y)-1\right]\right)\right]$$

now $\quad f_X(x) = p^x q \quad , \quad F_X(x) = 1 - p^{x+1}$

$\quad f_Y(y) = p^y q \quad , \quad F_Y(y) = 1 - p^{y+1}$

Therefore

$$R(k_o)=\sum_{x=k_o}^{\infty} \sum_{y=k_o}^{\infty} \left[p_1^x q_1 p_2^y q_2 \left(1+ v\left[2(1-p_1^{x+1}) - p_1^x q_1 - 1\right] \left[2(1-p_2^{y+1}) - p_2^y q_2 - 1\right]\right)\right]$$

$$= \sum_{x=k_o}^{\infty} p_1^x q_1 \sum_{y=k_o}^{\infty} p_2^y q_2 + v\left[\sum_{x=k_o}^{\infty} 2p_1^x q_1 - 2p_1^{2x+1} q_1 - p_1^{2x} q_1^2 - p_1^x q_1\right]$$

$$\left[\sum_{y=k_o}^{\infty} 2p_2^y q_2 - 2p_2^{2y+1} q_2 - p_2^{2y} q_2^2 - p_2^y q_2\right]$$

looking at this term by term

$$\sum_{x=k_o}^{\infty} p^x q = q \sum_{x=k_o}^{\infty} p^x = q\left[p^{k_o}+p^{k_o+1} + p^{k_o+2} + \cdots\right]$$

$$= q\, p^{k_o}\left[1 + p + p^2 + p^3 + \cdots\right]$$

$$= q\, p^{k_o} \sum_{i=0}^{\infty} p^i = q\, p^{k_o} \frac{1}{1 - p} = p^{k_o}$$

---

looking at

$$\sum_{x=k_o}^{\infty} \left[2p^x q - 2p^{2x+1} q - p^{2x} q^2 - p^x q\right]$$

$$= \sum_{x=k_o}^{\infty} p^x q - 2\sum_{x=k_o}^{\infty} p^{2x+1} q - \sum_{x=k_o}^{\infty} p^{2x} q^2$$

$$= p^{k_o} - 2pq \sum_{x=k_o}^{\infty} (p^2)^x - q^2 \sum_{x=k_o}^{\infty} (p^2)^x$$

$$= p^{k_o} - 2pq \frac{(p^2)^{k_o}}{1-p^2} - q^2 \frac{(p^2)^{k_o}}{1-p^2}$$

by putting (p^2) for p in $\sum_{x=k_o}^{\infty} p^x = \frac{p^{k_o}}{1-p}$

$$= p^{k_o} - \frac{2p\,(p^2)^{k_o}}{1+p} - \frac{q(p^2)^{k_o}}{1+p}$$

$$= \frac{p^{k_o}(1+p) - 2p^{2k_o+1} - p^{2k_o} + p^{2k_o+1}}{1 + p}$$

$$= \frac{p^{k_o}(1+p) - p^{2k_o} - p^{2k_o+1}}{1+p}$$

$$= \frac{(1+p)\, p^{k_o} - p^{2k_o}\,(1+p)}{1+p}$$

$$= p^{k_o} - p^{2k_o} = p^{k_o}(1 - p^{k_o})$$

Therefore putting these values back into equation for $R(k_o)$,

and using the corresponding terms for y, gives

$$R(k_o) = p_1^{k_o} p_2^{k_o} + v \left[\left[p_1^{k_o}(1 - p_1^{k_o})\right] \left[p_2^{k_o}(1 - p_2^{k_o})\right] \right]$$

If we call this $R_2(k_o)$ and call $R_1(k_o)$, the reliability obtained

when using the product rule, i.e.

$$R_1(k_o) = p_1^{k_o}\; p_2^{k_o}$$

then the difference, which we shall call $\Delta R(k_o)$ becomes

$$R(k_o) = R_2(k_o) - R_1(k_o)$$

$$= v \left[\left[p_1^{k_o}\,(1 - p_1^{k_o})\right] \left[p_2^{k_o}\,(1 - p_2^{k_o})\right] \right]$$

now putting it in terms of ρ gives

$$\Delta R(k_o) = \frac{(1+p_1)(1+p_2)}{\sqrt{p_1 \, p_2}} \rho \left[\left[p_1^{\,k_o}(1-p_1^{\,k_o}) \right] \left[p_2^{\,k_o}(1-p_2^{\,k_o}) \right] \right]$$

To determine the value of k_o at which $\Delta R(k_o)$ is maximum

$$\Delta R(k_o) = v \left[p_1^{\,k_o} - p_1^{\,2k_o} \right] \left[p_2^{\,k_o} - p_2^{\,2k_o} \right]$$

if we take the case $p_1 = p_2$ then

$$\Delta R(k_o) = v \left[p^{k_o} - p^{2k_o} \right]^2$$

we seek the derivative of this with respect to k_o

(v is independent of k_o)

$$\frac{d \; \Delta R(k_o)}{d \; k_o} = 2v \left[p^{k_o} - p^{2k_o} \right] \frac{d}{dk_o} \left[p^{k_o} - p^{2k_o} \right]$$

$$= 2 \; v \; (p^{k_o} - p^{2k_o})(p^{k_o} \ln p - 2p^{2k_o} \ln p)$$

$$= 2 \; v \; (p^{k_o} - p^{2k_o}) \ln p \; (p^{k_o} - 2p^{2k_o})$$

setting this equal to zero and solving for k_o

$$2 \; v \ln p \; (p^{k_o} - p^{2k_o})(p^{k_o} - 2p^{2k_o}) = 0$$

$$(p^{k_o} - p^{2k_o})(p^{k_o} - 2p^{2k_o}) = 0$$

one or both of these terms must be 0; if we take the first term

$$p^{k_o} - p^{2k_o} = 0$$

$$p^{k_o} = p^{2k_o}$$

$$k_o \ln p = 2\, k_o \ln p$$

$$k_o = 2\, k_o$$

$$\therefore k_o = 0$$

and it can be seen that this is a minimum point.

Now taking the second term

$$p^{k_o} - 2\, p^{2k_o} = 0$$

$$p^{k_o} = 2\, p^{2k_o}$$

$$k_o \ln p = \ln 2 + 2k_o \ln p$$

$$k_o (\ln p - 2 \ln p) = \ln 2$$

$$k_o = \frac{\ln 2}{\ln p - 2 \ln p} = - \frac{\ln 2}{\ln p}$$

hence

$$k_o = - \frac{\ln 2}{\ln p}$$

is a maximum point of $\Delta R(k_o)$ when $p_1 = p_2$.

APPENDIX A.3

MATHEMATICAL DEVELOPMENT

JOINT EXPONENTIAL, GEOMETRIC DISTRIBUTION

To show that $f_{XY}(x,y)$ is a probability function

when

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)\left[\, 1+v\ (2F_X(x)-1)(2F_Y(y)-1)\right]$$

where

$$f_X(x) = \frac{1}{a}\, e^{-x/a} \qquad 0 \leq x \qquad 0 < a \leq \infty$$

$$f_Y(y) = p^y q \qquad y = 0,\ 1,\ 2,\ \cdots\ ,\ 0<p<1$$

with $q = 1 - p$, $-1 \leq v \leq 1$

$$F_X(x) = \int_0^x f_X(x')dx' = 1 - e^{-x/a}$$

$$F_Y(y) = \sum_{y'=0}^{y} p^{y'} q = 1 - p^{y+1}$$

This gives

$$f_{XY}(x,y) = \frac{1}{a} e^{-x/a}\, p^y q \left[1 + v\left[2(1 - e^{-x/a}) - 1\right]\left[2(1 - p^{y+1}) - p^y q - 1\right]\right] \text{(B)}$$

Since $f_{XY}(x,y)$ is a continuous function over a countably infinite

number of points, i.e.

$$\int_{x+0}^{z} f_{XY}(x,y)\,dx \text{ has a value at integral values of } y \text{ only}$$

$$\int_{x=0}^{z} f_{XY}(x,y)\,dx = \begin{cases} \text{some positive value} & \text{if } y = 0, 1, 2, \cdots \\ 0 & \text{otherwise} \end{cases}$$

Then to evaluate $f_{XY}(x,y)$ over its entire domain we must look at

an infinite sum of such integrals.

To show that this $f_{XY}(x,y)$ is a probability function we must show

(1) $f_{XY}(x,y) \geq 0$

(2) $\sum_{y=0}^{\infty} \int_{x=0}^{\infty} f_{XY}(x,y) = 1$

(3) $\sum_{y=0}^{\infty} f_{XY}(x,y) = f_X(x)$, $\int_{x=0}^{\infty} f_{FY}(x,y)\,dx = f_Y(y)$

(1) using equation (B)

$$f_{XY}(x,y) = \frac{1}{a} e^{-x/a} p^y q \left[1+v \left[2(1 - e^{-x/a}) - 1 \right] \left[2(1 - p^{y+1}) - p^y q - 1 \right] \right]$$

$$= \frac{1}{a} e^{-x/a} p^y q \left[1+v \left[1 - 2e^{-x/a} \right] \left[1 - p^{y+1} - p^y \right] \right]$$

In Appendix A.1 it was shown that

$$-1 \leq 1 - 2e^{-x/a} \leq 1$$

and in Appendix A.2 that

$$-1 \leq 1 - p^{y+1} - p^y \leq 1$$

hence the quantity

$$1 + v \left[1 - 2e^{-x/a} \right] \left[1 - p^{y+1} - p^y \right] \geq 0$$

now since $\frac{1}{a} e^{-x/a} \geq 0$

and $p^y q \geq 0$

the product $\frac{1}{a} e^{-x/a} p^y q \geq 0$

Therefore

$$f_{XY}(x,y) \geq 0$$

---

(2) To show $\sum_{y=0}^{\infty} \int_{x=0}^{\infty} f_{XY}(x,y)\,dx = 1$

$$\sum_{y=0}^{\infty} \int_{x=0}^{\infty} f_{XY}(x,y)dx = \sum_{y=0}^{\infty} \int_{x=0}^{\infty} \left[\frac{1}{a} e^{-x/a} p^y q + v \left(e^{-x/a}(1-2e^{-x/a}) \right. \right.$$

$$\left. \left. p^y q(1-2p^{y+1}-p^y q) \right) \right] dx$$

$$= \sum_{y=0}^{\infty} p^y q \int_{x=0}^{\infty} \frac{1}{a} e^{-x/a} dx + v \left[\sum_{y=0}^{\infty} p^y q(1-2p^{y+1}-p^y q) \right.$$

$$\left. \int_{x=0}^{\infty} e^{-x/a}(1-2e^{-x/a})dx \right]$$

$$= (1)\ (1) + v\ [0 \cdot 0] = 1$$

calculations follow,

to show: $\int_{0}^{\infty} \frac{1}{a} e^{-x/a} dx = 1$

$$\int_{0}^{\infty} \frac{1}{a} e^{-x/a}\,dx = - \int_{0}^{\infty} e^{-x/a}(-\frac{1}{a}\,dx) = - e^{x/a} \Big]_{0}^{\infty} = - [0-1] = 1$$

to show: $$\sum_{y=0}^{\infty} p^{y} q = 1$$

$$\sum_{y=0}^{\infty} p^{y} q = q \sum_{y=0}^{\infty} p^{y}$$ geometric series which converges to

$$\frac{1}{1-p} \quad \text{if} \quad |p|<1$$

$$= q \frac{1}{1-p} = 1 \qquad \text{since} \quad q = 1 - p$$

to show: $$\int_{0}^{\infty} e^{-x/a}(1 - 2e^{-x/a})\, dx = 0$$

$$= \int_{0}^{\infty} e^{-x/a} dx - \int_{0}^{\infty} 2e^{-2x/a} dx$$

$$= -a \int_{0}^{\infty} e^{-x/a}\left(-\frac{1}{a}\, dx\right) - 2\left(-\frac{a}{2}\right) \int_{0}^{\infty} e^{-2x/a}\left(-\frac{2}{a}\, dx\right)$$

$$= -a\, e^{-x/a}\Big]_{0}^{\infty} + a\, e^{-2x/a}\Big]_{0}^{\infty} = -a\left[0-1\right] + a\left[0-1\right] = a-a = 0$$

to show: $\sum_{y=0}^{\infty} p^y q(1 - 2p^{y+1} - p^y q) = 0$

$$= \sum_{y=0}^{\infty} p^y q - 2 \sum_{y=0}^{\infty} p^{2y+1} q - \sum_{y=0}^{\infty} p^{2y} q^2$$

$$= q \frac{1}{1 - p} - 2pq \sum_{y=0}^{\infty} p^{2y} - q^2 \sum_{y=0}^{\infty} p^{2y}$$

$$= 1 - 2pq \sum_{y=0}^{\infty} (p^2)^y - q^2 \sum_{y=0}^{\infty} (p^2)^y$$

$$= 1 - 2pq \frac{1}{1 - p^2} - q^2 \frac{1}{1 - p^2} \qquad \text{since } p^2 < 1$$

$$= 1 - \frac{2pq}{(1-p)(1+p)} - \frac{q^2}{(1-p)(1+p)}$$

$$= 1 - \frac{2p}{1 + p} - \frac{q}{1 + p}$$

$$= 1 + \frac{-2p - (1 - p)}{1 + p}$$

$$= 1 + \frac{-2p - 1 + p}{1 + p}$$

$$= 1 + \frac{-p - 1}{1 + p}$$

$$= 1 - \frac{1 + p}{1 + p} = 1 - 1 = 0$$

(3) To show $\sum_{y=0}^{\infty} f_{XY}(x,y) = f_X(x)$

$$\sum_{y=0}^{\infty} f_{XY}(x,y) = \sum_{y=0}^{\infty} \frac{1}{a} e^{-x/a} p^y q \left[1+v\left[2(1-e^{-x/a}) - 1\right]\left[2(1-p^{y+1})-p^y q-1\right]\right]$$

$$= \frac{1}{a} e^{-x/a} \sum_{y=0}^{\infty} p^y q + v\left[e^{-x/a}(1-2e^{-x/a}\right]$$

$$\left[\sum_{y=0}^{\infty} p^y q(1 -2p^{y+1} - p^y q)\right]$$

but we have already shown, in Appendix A.2, that

$$\sum_{y=0}^{\infty} p^y q = 1$$

and that

$$\sum_{y=0}^{\infty} p^y q(1 - 2p^{y+1} - p^y q) = \sum_{y=0}^{\infty} \left[p^y q - 2p^{2y+1} q - p^{2y} q^2\right] = 0$$

hence

$$\sum_{y=0}^{\infty} f_{XY}(x,y) = \frac{1}{a} e^{-x/a}(1) + v\left[e^{-x/a}(1 - 2e^{-x/a})\right] \left[0\right]$$

$$= \frac{1}{a} e^{-x/a} = f_X(x)$$

To show $\displaystyle\int_{x=0}^{\infty} f_{XY}(x,y)dx = f_Y(y)$

$$\int_{x=0}^{\infty} f_{XY}(x,y)dx = \int_{x=0}^{\infty} \frac{1}{a} e^{-x/a} p^y q \Big[1+v \left[2(1-e^{-x/a})-1\right] \left[2(1-p^{y+1})-p^y q-1\right] \Big] dx$$

$$= p^y q \int_{x=0}^{\infty} \frac{1}{a} e^{-x/a} dx + v p^y q \left[2(1-p^{y+1})-p^y q-1\right] \int_{x=0}^{\infty} e^{-x/a}(1-2e^{-x/a})dx$$

but we have already shown, in Appendix A.1, that

$$\int_{x=0}^{\infty} \frac{1}{a} e^{-x/a} dx = 1$$

and that $\displaystyle\int_{x=0}^{\infty} e^{-x/a}(1 - 2e^{-x/a})dx = 0$

hence

$$\int_{x=0}^{\infty} f_{XY}(x,y)dx = p^y q(1)+v\, p^y q \left[2(1-p^{y+1})-p^y q-1\right] \left[0\right]$$

$$= p^y q = f_Y(y)$$

To calculate $E(X,Y)$ and ρ

Here we must again look at an infinite sum of integrals of $f_{XY}(x,y)$ weighted by the function xy.

$$E(x,y) = \sum_{y=0}^{\infty} \int_{x=0}^{\infty} xy \, f_{XY}(x,y)dx$$

from equation (B)

$$E(x,y) = \sum_{y=0}^{\infty} \int_{x=0}^{\infty} x \, \frac{1}{a} e^{-x/a} yp^y q \left[1+v \left[2(1-e^{-x/a})-1\right] \left[2(1-p^{y+1})-p^y q-1\right] \right] dx$$

$$E(x,y) = \sum_{y=0}^{\infty} yp^y q \int_{x=0}^{\infty} \frac{1}{a} x \, e^{-x/a}dx + v \left[\int_{0}^{\infty} \frac{1}{a} x \, e^{-x/a}(1-2e^{-x/a})dx \sum_{y=0}^{\infty} yp^y q(1-2p^{y+1} - p^y q) \right]$$

looking at $\displaystyle\int_{0}^{\infty} \frac{x}{a} e^{-x/a} dx$

using integration by parts: $\int u\,dv = uv - \int v\,du$

let: $u = \frac{x}{a}$ $\qquad dv = e^{-x/a}dx$

then $du = \frac{1}{a}\,dx$ $\qquad v = \int e^{-x/a}dx = -\,a\,e^{-x/a}$

$$\text{gives: } \int_0^\infty \frac{x}{a}\,e^{-x/a}\,dx = -\frac{x}{a}\,a\,e^{-x/a}\Big]_0^\infty - \int_0^\infty -\,a e^{-x/a}\,\frac{1}{a}\,dx$$

$$= -\,xe^{-x/a} - ae^{-x/a}\Big]_0^\infty = -\,e^{-x/a}(x+a)\Big]_0^\infty$$

$$-e^{-x/a}(x+a)\Big]_0^\infty = \lim_{B\to\infty}\Big|-e^{-x/a}(x+a)\Big|\Big]_0^B = \lim_{B\to\infty}\Big|-e^{-B/a}(B+a)+e^{o/a}(o+a)\Big|$$

$$= \lim_{B\to\infty}\left[a - e^{-B/a}(B+a)\right]$$

$$= \lim_{B\to\infty} a - \lim_{B\to\infty} B\,e^{-B/a} - \lim_{B\to\infty} a\,e^{-B/a}$$

$$= a - \lim_{B\to\infty}\frac{B}{e^{B/a}} - 0$$

$$= a - \lim_{B\to\infty}\frac{B}{e^{B/a}}$$

using L' Hopital's rule:

$$\lim_{B\to\infty} \frac{B}{e^{B/a}} = \lim_{B\to\infty} \frac{\frac{d(B)}{dB}}{\frac{d\, e^{B/a}}{dB}}$$

$$= \lim_{B\to\infty} \frac{1}{e^{B/a}} = 0$$

Therefore

$$\int_0^\infty \frac{x}{a} e^{-x/a} dx = a$$

---

looking at $\displaystyle\int_0^\infty \frac{x}{a} e^{-x/a} (1 - 2e^{-x/a}) dx$

$$= \int_0^\infty \frac{x}{a} e^{-x/a} dx - \int_0^\infty \frac{2x}{a} e^{-2x/a} dx$$

first integral is a (from above)

second integral: if we substitute $\frac{2}{a}$ for $\frac{1}{a}$ in above, we get

$$\int_0^\infty \frac{2x}{a} e^{-2x/a} dx = \frac{a}{2}$$

$$\therefore \int_0^\infty \frac{x}{a} e^{-x/a} dx - \int_0^\infty \frac{2x}{a} e^{-2x/a} dx = a - \frac{a}{2} = \frac{a}{2}$$

---

looking at $\sum_{y=0}^{\infty} y\ p^y q$

making use of Lemma 1, Appendix 2, we see that

$$\sum_{y=0}^{\infty} yp^y = \frac{p}{(1-p)^2} \qquad \text{since } |p| < 1$$

hence

$$\sum_{y=0}^{\infty} y\ p^y q = q \sum_{y=0}^{\infty} y\ p^y = q\ \frac{p}{(1-p)^2} = \frac{pq}{(1-p)(1-p)} = \frac{p}{1-p} = \frac{p}{q}$$

---

now look at $\sum_{y=0}^{\infty} (-2\ y\ p^{2y+1}\ q)$

$$= -2\ pq \sum_{y=0}^{\infty} y(p^2)^y = -\ 2\ pq\ \frac{p^2}{(1-p^2)^2} = \frac{-2\ p^3 q}{(1-p^2)(1-p^2)}$$

$$= \frac{-\ 2\ p^3\ q}{(1-p)(1+p)\ (1-p)(1+p)} = \frac{-\ 2p^3}{q(1+p)^2}$$

---

now look at

$$\sum_{y=0}^{\infty} (-y\ p^{2y} q^2) = -q^2 \sum_{y=0}^{\infty} y(p^2)^y = -q^2\ \frac{p^2}{(1-p^2)^2} = \frac{-\ q^2\ p^2}{(1-p)(1+p)(1-p)(1+p)}$$

$$= \frac{-\ p^2}{(1+p)^2}$$

Therefore $\sum_{y=0}^{\infty} y\, p^y q\, (1 - 2p^{y+1} - p^y q)$

$$= \sum_{y=0}^{\infty} y\, p^y q - 2\sum_{y=0}^{\infty} y\, p^{2y+1}\, q - \sum_{y=0}^{\infty} y\, p^{2y} q^2$$

$$= \frac{p}{q} - \frac{2p^3}{q(1+p)^2} - \frac{p^2}{(1+p)^2}$$

$$= \frac{p(1+p)^2 - 2p^3 - qp^2}{q(1+p)^2}$$

$$= \frac{p(1+2p+p^2) - 2p^3 - (1-p)p^2}{q(1+p)^2}$$

$$= \frac{p+2p^2 + p^3 - 2p^3 - p^2 + p^3}{q(1+p)^2}$$

$$= \frac{p^2+p}{q(1+p)^2} = \frac{p(1+p)}{q(1+p)^2} = \frac{p}{q(1+p)}$$

Therefore putting these values into equation for E(X, Y) gives

$$E(X, Y) = (a)\left(\frac{p}{q}\right) + v\left[\left(\frac{a}{2}\right)\left(\frac{p}{q(1+p)}\right)\right]$$

$$= \frac{a\,p}{q}\left[1 + \frac{v}{2(1+p)}\right]$$

To find expression for ρ

We have $E(X) = a, \; E(Y) = \dfrac{p}{q}, \; \sigma_X = a, \; \sigma_Y = \dfrac{\sqrt{p}}{q}$

and $$\rho = \frac{E(X,Y) - E(X)\,E(Y)}{\sigma_X \; \sigma_Y}$$

hence

$$\rho = \frac{\dfrac{a\,p}{q}\left(1 + \dfrac{v}{2(1+p)}\right) - \dfrac{a\,p}{q}}{a\,\dfrac{\sqrt{p}}{q}}$$

$$= \frac{p\left(1 + \dfrac{v}{2(1+p)} - 1\right)}{\sqrt{p}}$$

$$\rho = \frac{v\sqrt{p}}{2(1+p)}$$

$|v| \leq 1 \qquad 0 < p < 1$

therefore $|\rho| \leq \frac{1}{4}$

To evaluate $R(t,k)$ and determine $\Delta R(t,k)$

$$R(t,k) = P\left[X \geq t, \; Y \geq k\right] = \sum_{y=k}^{\infty} \int_{x=t}^{\infty} f_{XY}(x,y)$$

where again we must sum over y and integrate over x.

$$R(t,k) = \int_t^{\infty} \frac{1}{a} e^{-x/a} dx \sum_{y=k}^{\infty} p^y q + v\left[\int_t^{\infty} \frac{1}{a} e^{-x/a}(1-2e^{-x/a})dx \right.$$

$$\left. \sum_{y=k}^{\infty} p^y q \; (1-2p^{y+1}-p^y q)\right]$$

looking at

$$\int_t^{\infty} \frac{1}{a} e^{-x/a} dx = - e^{-x/a}\Big]_t^{\infty} = e^{-t/a}$$

---

looking at

$$\int_t^{\infty} \frac{1}{a} e^{-x/a}(1-2e^{-x/a})dx = \int_t^{\infty} \frac{1}{a} e^{-x/a} dx - \int_t^{\infty} \frac{2}{a} e^{-2x/a} dx$$

from above, first integral is $e^{-t/a}$, substituting $\frac{2}{a}$ for $\frac{1}{a}$ in above

integral gives $\int_t^{\infty} \frac{2}{a} e^{-2x/a} \, dx = e^{-2t/a}$

therefore $\int_t^{\infty} \frac{1}{a} e^{-x/a}(1-2e^{-x/a})dx = e^{-t/a} - e^{-2t/a} = e^{-t/a}(1-e^{-t/a})$

---

looking at

$$\sum_{y=k}^{\infty} p^y q = q \sum_{y=k}^{\infty} p^y = q\left[p^k + p^{k+1} + p^{k+2} + \cdots\right]$$

$$= q\, p^k \left[1 + p + p^2 + p^3 + \cdots\right]$$

$$= q\, p^k \, \frac{1}{1 - p}$$

$$= p^k$$

---

looking at

$$\sum_{y=k}^{\infty} p^y q(1 - 2p^{y+1} - p^y q)$$

$$= \sum_{y=k}^{\infty} p^y q - 2 \sum_{y=k}^{\infty} p^{2y+1} q - \sum_{y=k}^{\infty} p^{2y} q^2$$

$$= p^k - 2\, pq \sum_{y=k}^{\infty} (p^2)^y - q^2 \sum_{y=k}^{\infty} (p^2)^y$$

$$= p^k - 2pq \, \frac{(p^2)^k}{1 - p^2} - q^2 \, \frac{(p^2)^k}{1 - p^2}$$

$$= p^k - \frac{2p^{2k+1}}{1 + p} - \frac{(1 - p)\, p^{2k}}{1 + p}$$

$$= \frac{p^k + p^{k+1} - 2p^{2k+1} - p^{2k} + p^{2k+1}}{1 + p}$$

$$= \frac{p^k + p^{k+1} - p^{2k+1} - p^{2k}}{1 + p} = \frac{p^k(1+p) - p^{2k}(1+p)}{(1+p)}$$

$$= p^k - p^{2k} = p^k\,(1 - p^k)$$

Therefore putting these values into equation for $R(t,k)$ gives

$$R(t,k) = e^{-t/a}\,p^k + v\left[e^{-t/a}(1 - e^{-t/a}) \cdot p^k\,(1 - p^k)\right]$$

$$R(t,k) = e^{-t/a}\,p^k\left[1 + v\,(1 - e^{-t/a})(1 - p^k)\right]$$

calling this $R_2(t,k)$ and defining $R_1(t,k)$ as

$$R_1(t,k) = e^{-t/a}\,p^k \qquad \text{(product rule)}$$

Then if $\quad \Delta R(t,k) = R_2(t,k) - R_1(t,k)$

$$\Delta R(t,k) = v\,e^{-t/a}\,p^k\,(1 - e^{-t/a})(1 - p^k)$$

or in terms of ρ

$$\Delta R(t,k) = \frac{2\rho(1+p)}{\sqrt{p}}\left[e^{-t/a}p^k\,(1 - e^{-t/a})(1 - p^k)\right]$$